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THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF
THE MATHEMATICAL ASSOCIATION OF AMERICA, INC.

VOLUME 54 NUMBER 1

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JANUARY 1947

The AMERICAN MATHEMATICAL MONTHLY

(Founded in 1894 by Benjamin F. Finkel)

Carroll V. Newsom, *Editor*

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THIS IS THE OFFICIAL JOURNAL OF THE

MATHEMATICAL ASSOCIATION OF AMERICA, INC.

(Devoted to the Interests of Collegiate Mathematics)

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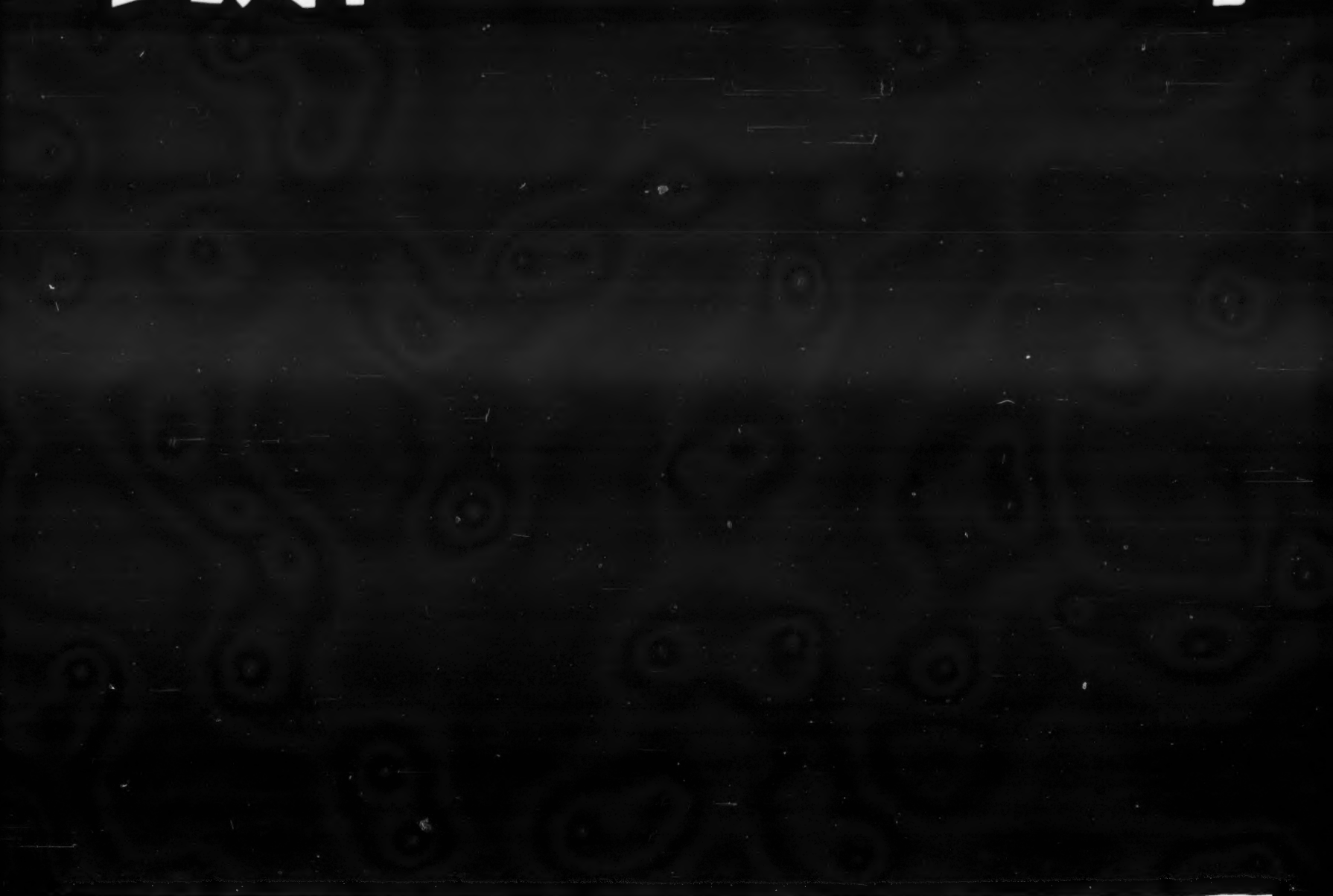
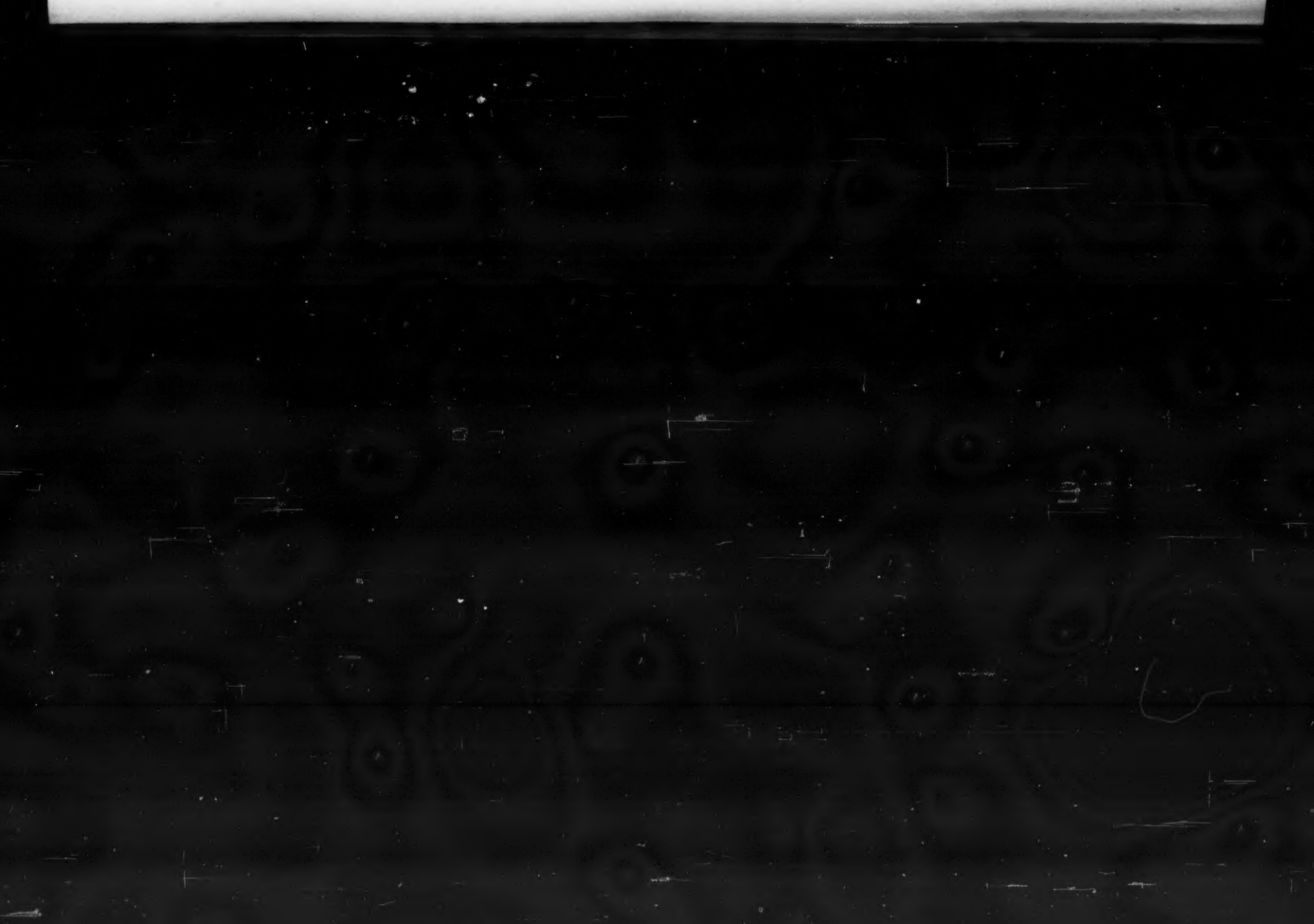
Entered as second class matter at the post office at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 538, P. L. and R., authorized April 1, 1926.

Subscription Price: To Members, $4 a Year, To Others, $5 a Year.

Published by the Association at Menasha, Wisconsin, and Oberlin, Ohio during the months of January, February, March, April, May, June-July, August-September, October, November, December.





COORDINATING HIGH SCHOOL AND COLLEGE MATHEMATICS*

W. D. REEVE, Teachers College, Columbia University

1. Introduction. The problem of coordinating high school and college mathematics is one which both the high school and college teachers of that subject should cooperate in solving. Failure to coordinate these separate fields in the past has led to a great deal of confusion and genuine loss both for the pupils involved and also their teachers.

In the first place, the high schools have not and still do not succeed in teaching their pupils as much as they are capable of learning if only they would wake up to their responsibilities in the matter. Two obvious reasons may be given for our present failure to teach more mathematics to those who are capable of learning it. They are both related to the same general problem of organization of content material.

2. Organization of subject matter. Our traditional water-tight compartment method of teaching algebra, then plane geometry, then intermediate algebra, and so on, leads to a great deal of unnecessary repetition of subject matter throughout the secondary school period that results in the loss of a great deal of time and energy. We teach algebra in the ninth grade, then plane geometry in the tenth grade. By that time the pupils have forgotten the algebra they knew so that when intermediate algebra (which should be a half year course as such) is presented, usually in the eleventh grade, a large part of the time is spent, if not wasted, in re-teaching ninth grade algebra. As a result it often happens that the entire eleventh year is spent on algebra when with better organization at least the ordinary course in trigonometry might have been presented. Then, if the pupil goes on with mathematics, he may be taught college algebra where again a great deal of time is spent, if not lost, re-teaching elementary and intermediate algebra. And as is often the case with such practice, very little college algebra *per se* is presented. I recently looked through a college algebra text in which well over a hundred pages were devoted to elementary algebra.

What is even worse so far as the pupils and even their teachers are concerned is for the pupils to have to take algebra, trigonometry or other really high school mathematics courses in the colleges and the universities for one reason or another and thus often miss analytic geometry and the calculus.

One excellent way to avoid all of this loss is to organize a course in general mathematics extending all through the high school so that those who can take it will have had a chance to study all of the mathematics up to and including the fundamental ideas of the calculus before they leave the high school. The college teachers of mathematics could then go on from there and devote their time to reorganizing what should really be mathematics on the college level and in that way make it possible for those boys and girls whose life work de-

* Presented to the Metropolitan New York Section of the Mathematical Association of America at Cooper Union, New York City, on May 4, 1946.

pends upon a strong background of mathematics to secure the necessary training without so much loss in time and energy.

The idea that general mathematics is a course for dullards keeps bobbing up in educational circles. I would like to point out here that this idea is not only unjustifiable, it is absolutely untrue. This view is no doubt due to the fact that those who so consider general mathematics do not really know what it means and that they have had little if any real experience in organizing and administering such a course.

If algebra were made the core in the ninth grade, geometry (plane and solid) in the tenth, algebra and trigonometry and simple analytics in the eleventh, analytics and the calculus in the twelfth and the necessary arithmetic worked in the course all along the line, it is demonstrable that not only much time could be saved, but the pupils concerned would have a much better background for their future work in mathematics and related fields.

The fact is that in a real important sense it takes a higher type of pupil to succeed in the advanced points of such a general course than an inferior one.

3. Individual differences. Failure to take account of individual difference in ability among high school pupils to say nothing of differences in interest and experiences, has resulted in great educational loss. The most retarded pupil in the secondary school today is the gifted pupil, the one with a scholarly mind. The secondary school machine is all geared up to turn out a mediocre product. This is obviously also true in the colleges. New objectives and new standards are needed. As Olson [1] recently put it:

"An even more fundamental reason for insisting upon standards of some kind in every type of education is the value of self-discipline. Whatever may be said of the unwisdom of disciplinary measures thrust on a person by someone else, there can be no doubt that success in any field is achieved only by those who are able to determine their course and then hold themselves to it. Whether we interpret education as life itself or merely as preparation for life, it must, in some way, furnish this same basic life situation. Much harm is being done in our schools and colleges by lowering standards to such a point that the students find no stimulus in their work. We talk bravely about challenging out students, yet we set our objectives so low that there is no challenge for most of them. The solution to this difficulty would be to group students according to their interests and abilities, and then set up appropriate standards which would challenge each group. Modern psychology tells us that virtually none of us succeeds in developing all his potentialities to the highest possible degree. That is painfully evident in our schools today. Many students with genuine ability go through college with poor work habits, largely because they early found they could make satisfactory grades with very little effort. Because the work was planned for the poorer students, there was little incentive for those with ability to do more. Undeveloped potentialities are surely as great an evil as a frustrating experience of failure."

Again we must not forget that for some minds in these days of mass education the kind of mathematics that the gifted pupils need is not the same for the slower type of mind. We must remember that in a democracy we are under great responsibility not only to train leaders, but also to develop *intelligent* followers as far as possible in the schools. We should develop at least a two-track if not three-track system to meet individual needs [2].

4. Reconciling points of view. Another difficulty that we face in trying to organize an adequate program of mathematics in the secondary school is the matter of reconciling points of view particularly among the general educationists and teachers of mathematics in the schools and colleges.

President James Bryant Conant of Harvard University has recently given an excellent suggestion. In discussing the reasons why the lay critics of secondary education talk as they do, President Conant said:

"I am almost tempted to generalize that the more educated the person, the less his knowledge of secondary-school education. Certainly the lack of knowledge among the professors of arts and sciences in our colleges and universities is proverbial. And with lack of information goes lack of understanding and lack of sympathy. As a result, on more than one campus we have almost a state of civil war between those who profess a knowledge of education and those who profess a knowledge of subjects which constitute a modern educational curriculum.

"This academic war has been in a sense inevitable, as I propose to show by a brief résumé of history, but to my mind an armistice has been for some years overdue. And it is for such an armistice that I should like to put in a good word this afternoon (and I might remark parenthetically that it takes two to make an armistice quite as much as to make a quarrel). My belief in the need for the cessation of hostilities comes not only from my general tendency to favor pacific methods of handling academic controversy, but also because I am really worried about the present lay reaction to educational matters. I am distressed by both the vehemence and the ignorance with which views about education are expressed publicly and privately by many prominent people. Now we can hardly expect the public to have a very clear understanding about educational problems when education is a house warring against itself. Hence my plea for a 'cease firing' order" [3].

This is the attitude that some of us have been taking for a long time. When teachers of secondary mathematics, college mathematics, supervisors, administrators and general educationists sit down around the table to discuss what for all of them should be a common problem, then we can hope for some practical solution of what should constitute general education in the post-war years.

We have a few outstanding cases* of men and women in the colleges and universities who have taken great interest in the work of the secondary schools

* Witness the great interest of men like Professors Cairns, Hedrick, Kasner, Mitchell, Slaught and others.

and in organizations of teachers in secondary schools, but for the most part many of these people do not know what is going on in the schools and some of them seem to care less. And this in spite of the fact that some of the pupils that are being trained in the mathematics of the secondary schools are the same pupils who will sit at the feet of the college instructors in mathematics in the colleges and universities in the days ahead. There is a great need for a better working understanding and cooperation between teachers of mathematics in the secondary school and those of collegiate grade.

5. What should be done. Let us now consider what should be done all along the line to improve the mathematical background of our people so that in peace time as well as in war time they can function as intelligent citizens whether or not a definite crisis exists.

In a recent editorial in *The National Mathematics Magazine* Professor S. T. Sanders said:

"Many teachers are nervously concerned over what may be the post-war status of school mathematics. The enormous expansion of the technical applications of the science under pressure of war has brought about a world-wide strengthening of mathematics in the school curriculum. Can this current academic primacy of mathematics be made permanent? Such is the question raised by those keenly mindful of the scant attention paid to this subject by the less recent curriculum makers.

"A careful study of the matter should not discount the fact that in respect to mathematics, the war has served only to bring about greatly multiplied *uses* of mathematics, a large proportion of which were already in existence. For, even in pre-war times, there had been for many years a steady growing public emphasis upon *applied* mathematics, rather than upon the logical or cultural aspects of the science.

"In the light of this definite trend, a trend not rooted in any war, it could well be that the post-war school effort should first be directed to discovering the mathematical aids or needs of all the major peacetime industrial enterprises. Cooperative programs initiated between industry and the schools would then have sounder foundations. Who shall say that the cultures of mathematics would be impaired by being stemmed in its utilities?" [4]

That great interest is being manifested in what place mathematics is to have in the schools after the war is evidenced by the many discussions that one hears, the various articles now appearing in current magazines and editorial comments like that above. Moreover, the National Council of Teachers of Mathematics has a *Commission on Post-War Plans* in mathematics, the first report of which appeared in *The Mathematics Teacher* for May, 1944. A second and more inclusive report of progress of this commission appeared in the May, 1945, issue of that journal.

The problem about which we are concerned here is that we do not have agreement in all quarters as to what should be done. We cannot take the space to

survey all of the recent opinions and articles, but a few typical ones will show how the wind is blowing.

After discussing the high place which mathematics once held in the schools and the poor results in mathematical education shown by recent Army reports, Professor H. L. Dorwart recently said:

"At this point, it may be asked why so many of our young people ceased to study mathematics some years ago. Many mathematics teachers say that it is all the fault of the educationists with their half-baked theories of the non-transfer of training and of the removal of everything difficult from education in order to prevent harmful personality development. The retort of the educationists is that mathematics teachers are just a lot of sadistic drill-masters who do more harm than good anyway. I will pass over this charge and countercharge in favor of what may, it is hoped, be a somewhat more helpful point of view.

"But, first, let us face the fact that there have been, and still are, both poor and poorly prepared teachers of mathematics, and that they have repelled many good students. Many high-school principals must be forced to give up the idea that anyone who possesses the credits in methods-of-education courses specified by law is thereby qualified to teach algebra or geometry. Also, college presidents and deans should investigate carefully the personalities of the instructors assigned to elementary courses. Even if the instructor is a recognized authority in his field, the conceited, arrogant, show-off type (fortunately few in number) should be used elsewhere than in elementary courses.

"I now propose the thesis that, once the requirements were withdrawn, students ceased to study mathematics principally because they did not recognize the fundamental rôle that it plays in modern civilization, or that by omitting the study of mathematics they were thereby imposing large restrictions on their future choice of profession or employment. In short, they did not recognize and usually have not been told that, in addition to serving, mathematics is a queen in her own right, a queen who will richly reward her followers, but only if they follow her diligently from their youth through a long period of time, even when the going is tough and when the path ahead is not always crystal-clear" [5].

In commenting on Dorwart's article, the late Professor Bagley said:

"The leading article in last week's number, *Mathematics—Queen and Handmaiden*, represents a type of discussion that is likely to become more and more important as the educational trends of the past few decades come increasingly to be scrutinized and questioned in the light of the educational weakness that the experiences of the war have revealed. Among these trends, probably none have been more debilitating than have the theories and arguments used to discount and discredit the subjects that are, to use a favorite phrase of the present writer, "exact and exacting, systematic and sequential"—more specifically, on the secondary level, mathematics and Latin.

"It is clear enough now that an educational development, fundamental in character and of far-reaching scope, lay back of the readiness with which

American education accepted theories and postulates that served to rationalize this discreditment. The upward expansion of mass education, which has resulted in what is virtually a "universal" high school, and which is now extending beyond the secondary level, would have been impossible without a relaxation of standards that are perhaps beyond the capacity and certainly alien to the tastes of a significant proportion of each generation of pupils and students.

"All this is water over the dam. The re-instating of Latin and of mathematics beyond the simple arithmetical process as secondary school requirements would doubtless be as unwise as it would be impossible. But it would be equally unwise, and it is not at all necessary, to continue allegiance to an educational theory that in effect encourages the following of the lines of least resistance by those who are competent to do the more difficult types of learning.

"Expert educational guidance, coupled with such masterful teaching as Dr. Dorwart suggests, can do much to solve the problem. But there should be as well, we think, a systematic, even a militant, effort to enlighten both youth and the public generally as to the basic significance of the subjects that constituted the core of liberal secondary education over the long period during which the secondary school was a highly selective institution. It was no more an accident that this core comprised Latin and mathematics than it was an accident that the core of elementary education from the outset comprised the reading and writing for the mother tongue and the primary arts of number. (After all, civilization itself began with the invention of writing and development of computation and measurement.) Liberal education on the secondary and higher levels has been based upon a refinement and expansion of the symbolism of conceptual thought. This has meant mathematics, and, in English-speaking countries, Latin, which, in a very real sense (as can be shown by abundant evidence), is the 'mother tongue of our mother tongue' in so far as the symbolism of advanced conceptual thought is concerned" [6].

Thus far the comments and articles referred to have been favorable to mathematics. However, we must present a typical point of view which raises questions as to whether mathematics is as important in the secondary school as some people think. Professor F. N. Freeman, Dean of the School of Education at the University of California, recently said:

"One of the outcomes of the war, in the opinion of many officers of the Army and Navy and of many observant laymen, is the revelation of gross inadequacy in the teaching of mathematics. The experience of the armed forces and of industry is supposed to have shown that vastly larger numbers of students should study mathematics and that they should study more mathematics—up to and including trigonometry. We may look, therefore, for a campaign after the war, to require the study of mathematics through trigonometry by a large share of high-school students. If such a campaign is launched, it will grow out of an idea that is about as sound as the belief that the psychological tests given in World War I showed 40% of the male population to be morons.

"The reasons that no such conclusion follows from the experience of the

Army and Navy, not to speak of industry, are, first, that the demands of the services in war time are no reflection of their demands in times of peace, and, second, that the number of men who need straight mathematics to perform duties required of them is much fewer than has been implied by the published statements. To say that the schools should, in peace time, give all the preparatory training that may be needed in time of war is like saying that industry must be kept in continual readiness to produce 10,000 planes a month. Again, the number of men who acquire techniques in the Army or Navy which are based on mathematics is vastly greater than the number who are required to understand the principles of mathematics and their application. The same is true of industry. The implied statement that all the men in the armed forces and industry who perform technical operations require a knowledge of higher mathematics for such performance is the wildest exaggeration. The war does not teach that mathematics through trigonometry is a practical necessity for a large proportion of men, let alone women. The question of the kind and amount of mathematics which is good for the average person and should be an element in general education still remains open" [7].

In the same article Dean Freeman presents a series of propositions which he thinks may be taken as a platform for the reorganization of mathematics in general education as follows:

"1. Only that mathematics is important for general education which the individual will use.
2. The individual will use only those mathematical ideas and operations which he has learned by use or in use.
3. The individual will actually use only those processes that he has mastered and made thoroughly familiar to himself.
4. Understanding is desirable, but it comes best through familiar use first, and formal explanation afterward.
5. Mathematics is, or contains, a form of language which formulates and defines the quantitative aspect of experience and, therefore, stimulates and largely creates quantitative ideas and forms of thought.
6. Mathematics may properly be thought of as a language—that is, as a particular set or particular sets of symbols which represent special aspects of reality.
7. To have meaning in the thinking of the child or of the ordinary person in general, the use of mathematical symbols and operations should be dedeveloped in intimate and continual association with the real world of experience.
8. The mathematics for the millions is that which gives clearer and more effective ways of thinking about the real world of experience because it has been developed out of this world of experience" [7].

I think Dean Freeman's points represent the thoughts of a great many powerful people in the education field.

It is gratifying to know that the Mathematical Association of America and

The National Council of Teachers of Mathematics are now cooperating in a plan which is intended to coordinate the studies of the two organizations in Mathematical Education. The Association has appointed a Committee under the chairmanship of Professor C. V. Newsom and the Council one under Professor F. L. Wren. Professor Newsom's committee has already issued a memorandum recommending that each Section of the Association give active attention to a number of points;* perhaps part of the time at each annual meeting could be devoted to a discussion of them. The Committee also recommended that each Section organize a committee to push the cause of mathematical education and to maintain contact with the central group.

6. Mathematics in an Air-Conditioned Civilization. In this complex civilization school administrators and teachers alike are faced with the problem of providing a better type of education for the seven million or so pupils who now make up our secondary school population. This means, not only drastic changes in the education of those pupils of the academic type to meet changing demands, but also an altogether different type of curriculum for those pupils who, under the traditional system, "leave school without skills or orientation of any kind with respect to social institutions" [8].

We have all heard the familiar remark, "We live in a changing world." The fact is that at the present time we not only live in a changing world, but the rate of change is so much more rapid than previously that few people are willing even to predict with any degree of precision just what will come to pass in the next fifteen or twenty years.

This generation and the one to follow are going to be "air conditioned," so to speak, and the sooner everybody adjusts his life to this situation the better for everybody concerned. The fact is that we must become air conditioned in order to survive.

The significance of all this for science in general and mathematics in particular is obvious to all who think seriously and fairly about the situation. These subjects, or at least their fundamental and useful aspects, should be *must* studies for most of our school population in the days to come.

Those who have been thinking about the matter already realize that in any well-organized course in air navigation many of the principles of mathematics, physics, and the other sciences previously taught in the secondary school can be applied in ways that will make these subjects interesting, meaningful, and more useful than ever before.

If we are to furnish an adequate basis for the education of pupils in the fundamental principles of aeronautics, then we shall have to get rid of the "deadwood" (the debris) that clutters up the courses in mathematics, physics, and other subjects in our secondary schools. Thus, in order to have time for some of the real life applications of mathematics to air navigation and other similarly im-

* Interested persons may obtain copies of this memorandum by writing to Professor C. V. Newsom, Oberlin College, Oberlin, Ohio.

portant life interests, we shall have to eliminate all such obsolete processes as certain types of factoring, complicated processes with polynomials, and the like, from the lower years of the secondary school and teach then in the later years only if and when they are needed. Moreover, this can be done without any loss to any important interest. But progress along this line is too slow. We need action and we need it quickly.

What groups of American citizens then are likely to need further training in mathematics in the days ahead and what type of organizations of subject matter and methods of instruction are likely to prove most efficient?

The purpose of a course in mathematics in the secondary school is to meet the needs of four groups of pupils:

1. Those who intend to go on to colleges and technical schools.
2. Those who are going to specialize in commercial work or vocational pursuits that require mathematics, especially algebra.
3. Those who intend to major in science.
4. Those who elect mathematics because they like it.

There is no question about the need for mathematical training for the pupils in the first group who are going to be engineers, statisticians, actuarial workers, certified public accountants and the like. The main problem here is one of guidance. If we can be certain that pupils are on the right track, then we can and should clear the way so that such pupils can be given not only four years of mathematics but mathematics of a very high order. To permit such pupils to take less mathematics and waste time in their courses by going at the rate of the less superior pupils will be tragic for them and the country as well. The same line of thought follows for those pupils who plan to go on to college where they will specialize in mathematics and then teach the subject, carry on research, or be applied mathematicians of one type or another.

A second group of skilled workers in business and industry will need to know certain mathematical facts and be able to use certain mathematical skills in the new era that will open up now that the war is over. Their work is extremely important, but they will not have to be as highly trained as those in the first group mentioned above. The possible future needs of such a group need to be studied carefully and a suitable course of study worked out to meet such needs.

A third group consists of those who intend to specialize in science. We ought to be able to assume that for them mathematical training of a high order would be basic if it were not for the fact that it does not work out that way. A careful study of the needs of this group should be made and the necessary content material organized for teaching purposes.

The last and by far the largest group of our citizens will be those who take mathematics because they are made to realize that without a reasonable understanding of the fundamental ideas they will not be competent to carry on as they should. The skills and habits of quantitative thinking must be acquired by such pupils if they are to meet the reasonable demands placed upon them.

In the new era just ahead many of our pupils will need to know how to solve

the problems of air and marine navigation. This will necessitate an understanding of the facts and processes of algebra, geometry and at least numerical trigonometry to an extent that will make the offering of at least a two-year course in mathematics starting with the ninth grade.

In the fields of business and industry where the war created new problems, different emphases may have to be made particularly where some of these problems of war are also the problems of the peace days ahead.

Let us also hope that more and better prepared teachers will be available and at salaries that will enable them to have a decent living. No matter what happens, however, we should not expect miracles and it is clear to those who understand the present situation that in many places better teachers, higher salaries, and more money for better books, equipment and the like may not be forthcoming.

In any case teachers who remain in the classroom because they want to do so in spite of handicaps must somehow learn to teach in better and more interesting ways many of the things that are now in the course of study. This need not be discouraging, for this can be done, if and only if, teachers are inspired to carry on the work as outlined by the commissions which have tried so hard to help.

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THE EULER COLUMN

W. B. CATON, University of Maine

1. Introduction. We shall consider the classical problem of the buckling of a uniform column under an axial load. For a detailed elementary discussion of this question, we refer the reader to Kármán and Biot **[1,** p. 295**]**. The reader may consult Love **[2,** Chap. 19**]** for a more advanced discussion of the problem under consideration.

We shall denote by a the length of the chord subtended by the central axis of the column, and the maximum deflexion of the central axis from its chord by d. The Taylor series expansion of the load P in terms of d^2/a^2 is obtained. Also, a similar expansion for P in terms of d^2/l^2 is found. Finally, estimates of the error committed in stopping with the p'th term of either series are found.

The actual formulas appearing in this note appertain to the first buckling mode only, but it is clear from the analysis that these results can be extended to the k'th mode at once. One has only to replace l and a wherever they appear in the formulas by l/k and a/k respectively.

2. Preliminary formulas and definitions. We consider a slender uniform column of length l, for which Young's modulus is E. Let I be the moment of inertia of the cross section about an axis through its centroid at right angles to the plane of bending. The lower end of the column is placed at the origin, and the central axis of the column is made to coincide with the x-axis taken vertically. The y-axis is taken horizontally, and directed to the right.

A load P, directed down along the x-axis, is then applied at the upper end of the column. The column buckles if P is sufficiently large, and the upper end descends along the x-axis to a point $(a, 0)$ at which equilibrium is attained. It is supposed that the lower end remains at $(0, 0)$, and that no restraints are placed on the slope of the central axis at either end.

The above considerations together with the differential equation of the elastic curve,

$$\frac{d^2y}{dx^2}\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{-3/2} = -\frac{Py}{EI}, \tag{1}$$

imply

$$l = 2\sqrt{\frac{EI}{P}}\,K(\pi/2,\, k), \tag{2}$$

$$a = 2\sqrt{\frac{EI}{P}}\,\{2E(\pi/2,\, k) - K(\pi/2,\, k)\}, \tag{3}$$

$$k^2 = Pd^2/4EI.$$

In the display above, $K(\pi/2, k)$ and $E(\pi/2, k)$ denote the complete elliptic integrals of the first and second kinds respectively.

Formulas (2) and (3) appear in Love [2, p. 403], but it is believed that some observations on their proofs would be of interest to readers of this note. Proofs of (2) and (3) will be sketched in section 3.

Elementary considerations show that $k^2 \leqq 1/2$ if the angle between the tangent at the origin and the chord is less than or equal to 90°.

It is perhaps worth while to notice in passing that (2) and (3) imply

$$\frac{l-a}{d} = \frac{2}{k}\{K(\pi/2, k) - E(\pi/2, k)\},$$

and this in turn gives

$$l - a = \frac{\pi d^2}{4}\sqrt{\frac{P}{EI}}\{1 + O(k^2)\}.$$

We set $w = k^2$ and put

$$K_1(w) = K(\pi/2, k), \qquad E_1(w) = E(\pi/2, k).$$

Also, we introduce

$$K^*(w) = \{K_1(w)\}^2, \qquad E^*(w) = \{2E_1(w) - K_1(w)\}^2.$$

Next, we define r_1 and r_2 as follows:

$$r_1 = \min_{0<R<1}\left\{\frac{1}{R}\max_{|\zeta|=R}|K^*(\zeta)|\right\}, \tag{4}$$

$$r_2 = \min_{0<R<1}\left\{\frac{1}{R}\max_{|\zeta|=R}|E^*(\zeta)|\right\}. \tag{5}$$

In the definitions of r_1 and r_2, ζ denotes a complex variable and $|\zeta| = R$ is the circle in the ζ plane with center at the origin and radius R. We may suppose that r_1 and r_2 are taken on for $R = R_1$ and R_2 respectively. A simple argument, based on the principle of the maximum, shows that R_1 and R_2 are unique.

3. On formulas (2) and (3).

We set

$$\frac{dy}{dx} = p, \qquad \frac{d^2y}{dx^2} = p\frac{dp}{dy},$$

in (1), and after integration, obtain

$$(1 + p^2)^{-1/2} = \frac{Py^2}{2EI} + c_1, \tag{6}$$

where c_1 is the constant of integration.

We now notice that since the beam is uniform, and the buckling mode is of

the first type, $x = a/2$ implies $p = 0$ and $y = d$, where d is the maximum deflexion. Hence,

$$c_1 = 1 - Pd^2/2EI = 1 - 2k^2. \tag{7}$$

Next, (6), (7), and the usual formula for arc-length from the calculus, gives

$$l/2 = \int_0^d \left\{1 - \left(c_1 + \frac{Py^2}{2EI}\right)^2\right\}^{-1/2} dy. \tag{8}$$

Introduce $k' > 0$,

$$k^2 + k'^2 = 1.$$

Formula (8) may now be put in the form

$$l/2 = \sqrt{\frac{\overline{EI}}{P}} \int_0^{1/k'} \{(1 + k^2u^2)(1 - k'^2u^2)\}^{-1/2} dy.$$

Now from (3, p. 494),

$$l = 2\sqrt{\frac{\overline{EI}}{P}}\, sd^{-1}(1/k'),$$

which implies (2). In the notation of Glaisher (explained in *op. cit.*), $sd^{-1}(u)$ is the inverse of the elliptic function

$$sd(u) \equiv \frac{sn(u)}{dn(u)}.$$

The proof of (3) is much the same. From (6) we have

$$\left(c_1 + \frac{Py^2}{2EI}\right)\left\{1 - \left(c_1 + \frac{Py^2}{2EI}\right)^2\right\}^{-1/2} dy = dx,$$

with the value of c_1 obtained from (7). Integrating the right hand member between 0 and $a/2$, and the left hand member between the corresponding limits 0 and d, we obtain with the help of (8),

$$\frac{c_1 l}{2} + \frac{P}{2EI} \int_0^d \left\{1 - \left(c_1 + \frac{Py^2}{2EI}\right)^2\right\}^{-1/2} y^2 dy = a/2. \tag{9}$$

Formula (9) is next put up in the form

$$\frac{c_1 l}{2} + 2k^2k'^2 \sqrt{\frac{\overline{EI}}{P}} \int_0^d \{(1 + k^2u^2)(1 - k'^2u^2)\}^{-1/2} u^2 du = a/2. \tag{10}$$

By means of the transformation

$$u = sd(v),$$

the integral in (10) can be reduced to

$$\int_{l_1}^{l_2} sd^2(v)dv, \tag{11}$$

with

$$l_1 = sd^{-1}(0) = 0, l_2 = sd^{-1}(1/k') = K(\pi/2, k) \equiv K.$$

Now, integral (11) is well known, and we have

$$k^2k'^2\int_0^K sd^2(v)dv = E(\pi/2, k) - k^2K(\pi/2, k). \tag{12}$$

Formula (12) follows from a result in Whittaker and Watson [3, p. 516].

Finally, substitute the value of the integral obtained in (12) back in (10) to get (3).

4. Main results and their proofs.

THEOREM 1. *If* $w = Pd^2/4EI$, *then*

$$w = \sum_1^\infty \frac{t^n}{n!}\left[\frac{d^{n-1}}{d\zeta^{n-1}}\{K^*(\zeta)\}^n\right]_{\zeta=0},$$

valid for $t = d^2/l^2 < 1/r_1$.

The remainder after p terms of the series in Theorem 1 is not greater than

$$-R_1(tr_1)^p \log(1 - tr_1)$$

if $tr_1 < 1$.

THEOREM 2. *If* $w = Pd^2/4EI$, *then*

$$w = \sum_1^\infty \frac{r^n}{n!}\left[\frac{d^{n-1}}{d\zeta^{n-1}}\{E^*(\zeta)\}^n\right]_{\zeta=0},$$

valid for $r = d^2/a^2 < 1/r_2$.

An estimate of the remainder after p terms of the type obtained for the series in Theorem 1 holds here also.

COROLLARY 1.

$$P = \frac{\pi^2EI}{l^2}\left\{1 + \frac{\pi^2d^2}{2^3l^2} + \frac{19\pi^4d^4}{2^9l^4} + \cdots\right\},$$

valid for $d^2/l^2 < 1/r_1$.

COROLLARY 2.

$$P = \frac{\pi^2EI}{a^2}\left\{1 - \frac{3\pi^2d^2}{2^3a^2} + \cdots\right\},$$

valid for $d^2/a^2<1/r_2$.

COROLLARY 3.

$$\left|P-\frac{\pi^2EI}{l^2}\right|<-\frac{4EIR_1r_1}{l^2}\log\left(1-\frac{d^2r_1}{l^2}\right),$$

if $d^2/l^2<1/r_2$.

We observe that from d/l or d/a one could compute P from Corollary 1 or 2. Notice too that in Corollary 3 we have an estimate of the error committed when one uses the common engineering formula,

$$P\cong\frac{\pi^2EI}{l^2}.$$

The proof of Theorem 1 will now be sketched, that of Theorem 2 is much the same, so will be omitted.

Equation (2) implies that $\zeta=w$ is a root of

$$\zeta=tK^*(\zeta).$$

The Lagrange expansion theorem, (see Whittaker and Watson [3, p. 132]), is now applied. We take the $f(\zeta)$ of the theorem to be ζ and $a=0$. Also, as the integration contour we choose

$$|\zeta|=R_1.$$

Then, on this contour

$$\left|\frac{K^*(\zeta)}{\zeta}\right|\leqq r_1,$$

and this implies

$$t|K^*(\zeta)|<|\zeta|,$$

for any ζ on the contour $|\zeta|=R_1$, and any $t<1/r_1$.

Thus, the conditions of the Lagrange theorem are satisfied and so we get the expansion of Theorem 1 at once.

For the estimate of the remainder, we notice that the Cauchy integral formula applied to

$$\frac{1}{(n-1)!}\left[\frac{d^{n-1}}{d\zeta^{n-1}}\{K^*(\zeta)\}^n\right]_{\zeta=0},$$

shows that the remainder of the series in Theorem 1 after p terms is not greater than

$$R_1\sum_{n=p+1}^{\infty}\frac{(tr_1)^n}{n} \tag{13}$$

if $tr_1 < 1$. From (13), the estimate given follows at once.

For the proof of Theorem 2, we start with (3), and it implies $\zeta = w$ is a root of

$$\zeta = rE^*(\zeta).$$

The rest of the proof is the same as above.

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CONJUGO-CONJUGATE COUPLES IN INVOLUTION

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1. Introduction. That a certain type of involution of points on a line, namely, the hyperbolic, has imaginary elements is a fact well known and long known. [1] Geometers in general have been inclined to "bypass" this interesting assemblage going on to the higher and wider aspects of the subject of involution. Since, however, this assemblage of imaginary elements is infinite, it seems that it deserves further consideration and explication, and such is the object of this paper.

It not infrequently happens when a problem of geometry has its origin on the algebraical side of analysis, or when the algebra is called into service, complete generality being sought, that the imaginary obtrudes itself demanding consideration. This, too, notwithstanding the fact that the constants and parameters employed are all real. Such is the case of the involution of points on a line generated, say, by a pencil of quadratics, the roots of the pencil being, of course, the coordinates of the conjugate point-pairs of the involution, [2] or, and probably, more directly the result of normalizing [3] or polarizing [4] a single quadratic, the roots of the quadratic being the double points of the involution. We shall incline rather to this latter approach.

It turns out, as we shall see in this paper, that in the involution of points on a line, real constants being employed, there exists a single infinitude of point-pairs which on the one hand are conjugate harmonically with respect to the double points, and on the other hand are conjugate imaginaries. The object of this paper, therefore, is primarily: To "locate" this infinite assemblage of imaginaries, and to give them a "real" representation; secondarily: To give a complete characterization of the type of involution (hyperbolic) in which these elements appear; and finally: To give a complete set of constructions for finding the double-points of the involution determined by any two given point-pairs.

2. Definition. Because of the two-fold bond (yoke) pairing the elements of this infinite assemblage—conjugate harmonically and conjugate imaginary—we shall refer to them as the conjugo-conjugate couples in involution. In speaking

of point-pairs in general in the remainder of this paper, when the pairing is effected through an involution, we shall employ the term couple since it has a stronger connotation than that of point-pair. The involution is, of course, taken on a line, and by line we shall mean a straight line having a real trace but endowed with imaginary as well as real points. The constants (coefficients and parameters) employed shall be real numbers. The line on which the involution lies we shall call the base line, and the circle centered on the center of the involution and passing through the double-points we shall call the base circle. The double points we shall refer to as doubles.

3. Generation. The symmetrical lineo-linear relation:

$$ax_1x_2 + b(x_1 + x_2) + c = 0 \tag{1}$$

defines an involution of points on a line if we agree that the numbers x_1 and x_2 shall be the coordinates of points, X_1 and X_2 say, referred to some convenient origin: $X_1(x_1)$, $X_2(x_2)$. Given any point $X(x)$ on the base line, its mate is given by (1) and together they constitute an involutory couple. There are two points each of which is its own mate. Together they constitute the doubles of the involution. These doubles are determined by $x_1 = x_2$ and are furnished by

$$ax^2 + 2bx + c = 0. \tag{2}$$

If we represent the doubles by $D_1(d_1)$ and $D_2(d_2)$ then:

$$d_1 \mid d_2 = (-b + \mid -\sqrt{\Delta})/a \quad \text{where} \quad \Delta = b^2 - ac. \tag{3}$$

The type of double and in turn the type of involution is determined by Δ. For $\Delta > 0$ the doubles are real and distinct and the involution is hyperbolic; for $\Delta < 0$ the doubles are conjugate imaginary and the involution is elliptic; for $\Delta = 0$, the doubles double up, and the involution is parabolic.

Each real point of the base line has a real harmonic conjugate with respect to the doubles. Together they constitute a real couple, and we may pair them ad libitum by assuming real values for x_1 and calculating the corresponding value of x_2 employing (1). This furnishes a single infinitude of real couples which are harmonic conjugates with respect to the double points.

We discover an enlarged situation, however, when we pose the question: Is every point on the base line the mid-point of a couple? The answer is: Yes, but the couples are not always real. Let us call the mid-point of a couple $M(\mu)$ and let δ be the digression of the individuals of the couple from M; that is, the individual points of the couple have for coordinates $\mu+\delta$ and $\mu-\delta$: $X_1(\mu+\delta)$, $X_2(\mu-\delta)$. Then we shall have from (1)

$$a(\mu^2 - \delta^2) + 2b\mu + c = 0, \quad \text{or}$$

$$\delta^2 = \left(\mu + \frac{b}{a}\right)^2 - \frac{\Delta}{a^2}, \quad \text{or} \tag{4}$$

$$\delta^2 = (\mu - d_1)(\mu - d_2). \tag{5}$$

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From (4) we see that δ^2 is positive whenever $\Delta \leqq 0$. These are the elliptic and parabolic types. Hence, neither type has conjugo-conjugate couples. When $\Delta > 0$ there is the possibility of δ^2 negative for certain values of μ. If we consider $d_1 < d_2$ we see from (5) that when $\mu < d_1$ and when $\mu > d_2$, δ^2 is positive. That is, when M is without the segment D_1D_2 or outside the base circle, it is the mid-point of a real couple. When, however, $d_1 > \mu > d_2$ the first factor of the second member of (5) is positive and the second is negative; hence δ^2 is negative and δ is pure imaginary, and therefore the corresponding couples are conjugate imaginary. This situation occurs only in the hyperbolic type of involution [5].

The geometry may be called into case here to explicate the situation still further. The equation of the base circle is

$$a(x^2 + y^2) + 2bx + c = 0$$

and the equation of the congruence of circles orthogonal thereto is

$$a(x^2 + y^2) - 2a\mu x - 2a\nu y - 2b\mu - c = 0.$$

The parameters are μ and ν, μ being the coordinate of the mid-point M of a conjugate couple; for, letting $y = 0$ and solving for x we have x_1, $x_2 = \mu \pm \delta$, as we should. The congruence may be written in the form

$$(x - \mu)^2 + (y - \nu)^2 = \nu^2 + \delta^2;$$

hence (μ, ν) is the center of the circle of the congruence and $\sqrt{\nu^2+\delta^2}$ is its radius. Now, as we have seen, there are values of μ which make δ^2 negative, namely $d_1 > \mu > d_2$. For all such values of μ the radius of the congruence is less than ν the ordinate of the center of the circle.

Let us construct the lines $x = d_1$ and $x = d_2$ (Fig. 2). Then every circle having its center without the strip determined by these lines will have its radius greater than ν and will cut the base line in a real conjugate couple as (1); every circle having its center on one of these lines will have a radius equal to ν and will cut the base line in one of the doubles as (2); and, finally, every circle having its center within the strip will have a radius less than ν and will cut the base line in a conjugo-conjugate couple as (3). Thus, the mid-points of our conjugo-conjugate couples all lie within the strip determined by two lines through the double points perpendicular to the base line.

4. Location. Stated geometrically, then, we have the following theorem:

Every point on the base line of an involution of points is the mid-point of a conjugate couple; if the point is without the base circle, the couple is real, and there is a single infinitude of them; if the point is on the base circle, the individuals of the couple coalesce, and we have the doubles; if the point is within the base circle, the couple is (conjugate) imaginary, and there is a single infinitude of them.

These latter are our conjugo-conjugate couples in a quadratic involution of points on a line, and thus are they "located."

5. Representation. We have succeeded in "locating" our conjugo-conjugate couples in the involution of points on a line. We propose now to give them a (real) representation. The method of representing imaginary configurations by real ones employed by Laguerre lends itself both readily and adequately. We make a brief statement of the method omitting the proofs [6].

The complex projective plane is four-dimensional (∞^4) in complex points, the term complex comprehending both the real and the imaginary. In this quadruple infinitude there is a sub-assemblage consisting of a double infinitude (∞^2) of real points constituting the real (projective) plane. The imaginary points of the quadruple infinitude are paired conjugately. The multiplicity of points in the complex plane is co-dimensional with the multiplicity of point-pairs in the real plane; so that if we had a scheme setting up a one-to-one relationship (transformation) pairing each (finite) point of the complex plane with each (finite) point-pair of the real plane and vice versa, then we should have a (real) representation of the imaginary points in the complex plane. This is exactly what the Laguerre representation does, according to the following free translation from the French:

(a) "Let α be an imaginary point of a plane; through this point pass an isotropic of the first system having one real point A and an isotropic of the second system having one real point A'. It is clear that these two points are completely determined by the point α, and that reciprocally α is determined without ambiguity by the points A and A'.
"I shall say that AA' is the representative segment of the point α, A being the initial point and A' the terminal point of this segment."

(b) "If AA' represents an imaginary point α, then $A'A$ represents its conjugate $\bar{\alpha}$."

(c) "The real line which joins two conjugate imaginary points α, $\bar{\alpha}$ is the perpendicular bisector of their representative segment AA'."

(d) "Having given a real circle, in order that a segment AA' may represent a point α on this circle, it is necessary and sufficient that the points A and A' be mutually inverse in respect to this circle." [7]

Thus, in the light of the first statement of Laguerre, if A and $\overline{A}$ are two conjugate imaginary points and if P and P' are the initial and terminal points of the real ordered point-pair which constitutes the real representation of $A:A(P\rightarrow P')$, then reversing the order we have the real representation of $\overline{A}$, the conjugate of $A:\overline{A}(P'\rightarrow P)$. We shall employ this notation in the designation of an imaginary point and its (real) representation.

From the third statement we see that the (real) representation of the complex line with a real trace is the ensemble of ordered point-pairs which are symmetrical, the one the reflection of the other, with respect to the real trace. And again from the fourth statement we have it that the (real) representation of the complex circle having a real trace is the assemblage of ordered point-pairs which are mutually inverse with respect to the real trace. Both the complex line and the complex circle are two-dimensional (∞^2) in complex points. Then every point

of the finite projective real plane is both the initial point and the terminal point in the representation of complex points on the line and the circle. Hence to find the representation of their intersection when their traces do not meet is to find that ordered point-pair which they have in common.

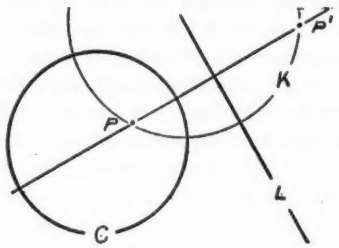

FIG. 1

Thus, in Figure 1, to find the (real) representation of the intersection of the line L and the circle (C) we seek that ordered point-pair which is common to both. By reason of the symmetry involved we see that the required point-pair will lie on a line through the center of the circle perpendicular to the line. The pair is discovered when we cut this line with a circle centered on L and orthogonal to (C) as for example (K). Hence if A and $\overline{A}$ are the intersections of L and (C), then $A(P \rightarrow P')$ and $\overline{A}(P' \rightarrow P)$.

Let us return now to our problem. The parameter ν is really redundant, the problem does not really require a congruence of circles, but only a pencil. We may choose this pencil out of our congruence in a variety of ways: circles having a common point, having a common radius, centered on a line oblique to the base line. We shall choose that pencil of circles for which ν is constant, say ν_0. Any finite value chosen for ν_0 is adequate but if we choose $\nu_0 < \Delta$ some of the circles of the pencil will be imaginary, that is, they will have imaginary radii and hence will not have real traces. This case is taken care of by the Laguerre representation, of course, but introduces an unnecessary element into the problem which we shall side step by taking $\nu_0 > \Delta$.

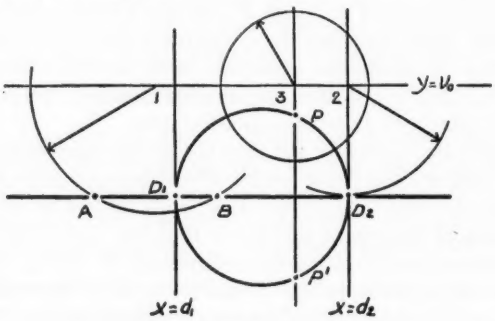

FIG. 2

Referring now to Figure 2, the line of centers of the pencil of circles which cuts the involution under consideration from the base line is the line $y=\nu_0$, $\nu_0>\Delta$. The centers of the circles then have for coordinates (μ, ν_0). When the center lies without the strip $x=d_1$, $x=d_2$, the circle will cut out a real conjugate couple, as circle (1). When the center is on one of the lines $x=d_1$ or $x=d_2$, the circle is tangent to the base line, and hence cuts it in a double as circle (2). When the center of the circle lies within the strip, the circle cuts the base line in one of our conjugo-conjugate couples, as circle (3). Our problem is to find the real representation of this conjugo-conjugate couple. Referring to Figure 1 we see at once that it is the point-pair $P \rightleftarrows P'$. That is, if the conjugo-conjugate couple is A, $\overline{A}$, then $A(P \rightarrow P')$ and $\overline{A}(P' \rightarrow P)$. There is a single infinitude of such points and their (real) representation is the assemblage of ordered point-pairs on the base circle which are symmetric (the one the reflection of the other) with respect to the base line.

6. Characterization. We may now give a complete characterization of the hyperbolic type of involution on a line having a real trace, real quantities only being employed in its generation.

The double points are real. The conjugate point-pairs consist on the one hand of a single infinitude of real pairs whose mid-points lie without the segment determined by the double points; and, on the other hand, of a single infinitude of imaginary pairs whose mid-points lie within the segment determined by the double points.

If we agree to speak of the real representation of the imaginary pairs as of the pairs themselves, and that is the function of a representative; and if we further agree to speak of reflection in a line as an inversion, and that is really what it is; then we may say:

The hyperbolic involution of points on a line with a real trace consists of a single infinitude of real point-pairs and a single infinitude of imaginary point-pairs. The real point-pairs are those points on the base line which are mutually inverse with respect to the base circle, while the imaginary point-pairs are those points on the base circle which are mutually inverse with respect to the base line.

7. Illustration. The Apollonian Circles of a triangle determine an elliptic involution of circles orthogonal to the circumcircle. [8] This pencil of circles cuts from any diameter of the circumcircle a hyperbolic involution of points. In particular it cuts such an involution from the Euler line E. The line of centers of the Apollonian Circles is the Lemoine Line L. All the circles of the pencil whose centers lie within the segment determined by the tangents to the circumcircle perpendicular to the Euler Line will cut from the Euler Line the conjugo-conjugate couples with which this paper is concerned. One of the Apollonian Circles in Figure 3 is such a circle. Incidentally, the common points of the Apollonian Circles are the Isodynamic Points, I, I' of the triangle whose properties are noted by Johnson, and Altschiller-Court. [8] In addition, we call attention to the fact that when ordered they constitute the real representation of the intersection of the Lemoine Line and the circumcircle. Again they are the

real representation of the double points of the elliptic involution determined by the Apollonian Circles on the Lemoine Line.*

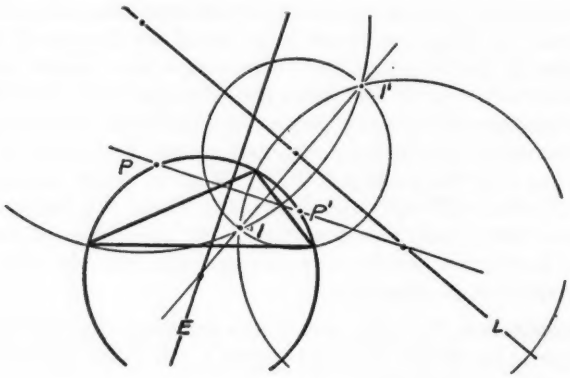

FIG. 3

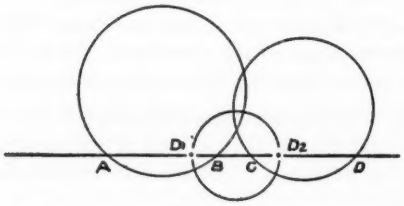

FIG. 4

8. Construction. We conclude with a problem of construction: Given two point-pairs on a line to construct the involution, *i.e.* find the double points, determined thereby. The point-pairs may not interlace and they may not share a mutual individual point since these determine the elliptic and parabolic types respectively. Thus we recognize three cases:

1. Both point-pairs real (and noninterlaced)
2. One pair real, one conjugate imaginary
3. Both pairs conjugate imaginary.

The construction is the same for all three cases: (a) Construct a circle on each point-pair, (b) construct the radical axis (so called) of the circles in (a); this line determines the center of involution, (c) construct the circle centered on the center of involution and orthogonal to the circles in (a). This circle cuts the base line in the required double points.

* We have employed this configuration merely as an illustration, but it seems to deserve consideration per se: for it is a mutual relationship between the Lemoine Line and the Euler Line and this mutuality extends to the given triangle and a certain triangle of which the Lemoine Line is the Euler Line and the Euler Line of the original triangle the Lemoine Line.

Case 1. Figure 4. Given the point-pairs A, B; C, D.

Case 2. Figure 5. Given the point-pairs A, B; C, $\overline{C}$; $C(P\rightarrow P')$, $\overline{C}(P'\rightarrow P)$.

Case 3. Figure 6. Given the point-pairs A, $\overline{A}$; B, $\overline{B}$; $A(Q\rightarrow Q')$, $\overline{A}(Q'\rightarrow Q)$; $B(P\rightarrow P')$, $\overline{B}(P'\rightarrow P)$.

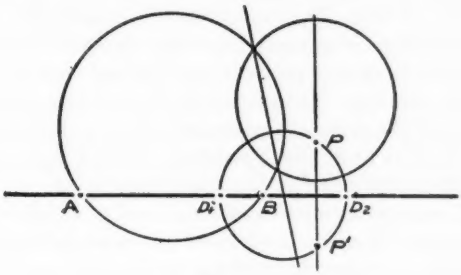

FIG. 5

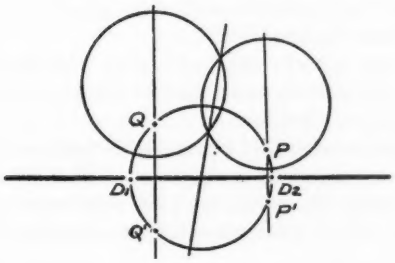

FIG. 6

If one pair is a double point, no modification of the method is necessary to secure the other double point.

The subject of involution on a complex line and its accompanying problems of construction has been generalized by the author and may merit publication in the future [9].

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7. Laguerre, Oeuvres, Tome II (a) p. 352; (b) and (c) p. 94; (d) p. 103. Reference should also be made to Study, Vorlesungen über Ausgewählte Gegenstände der Geometrie, erstes Heft, 1911, pp. 8–10, 25–32; and to Coolidge, The Geometry of the Complex Domain, 1924, pp. 85–91.
8. Johnson, Modern Geometry, Sec. 490 et seq. Altschiller-Court, College Geometry, Sec. 466 et seq.
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THE JEEP PROBLEM

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1. Introduction. The problem in logistics with which this paper deals was proposed to the author by Gail Young and Ivan Niven, both of Purdue University, in the latter part of 1945. The original source is unknown to the author. At that time Niven had obtained a partial solution based on certain assumptions. After the first submission of this paper it was learned that L. Alaoglu had also obtained a complete solution. He mentioned that similar problems had arisen in air transport operations in the China theater. It has also been suggested that there may be applications to Arctic expeditions and interplanetary travel. This paper, however, will confine itself to the lowly jeep.

Suppose that a jeep can carry a maximum load of n gallons of gas and can travel c miles per gallon. The jeep is required to cross a desert x miles wide. Our problem is to prescribe a method for making the journey most economically and to find the least sufficient amount of gas. It is not obvious that such a method exists, and it would be more exact to speak of the greatest lower bound, until the existence of the minimum is established.

We shall assume that n and c are both unity. This involves no loss of generality; it is equivalent to taking as our unit of distance nc, the number of miles that the jeep can travel on a full load.

If $x \leqq 1$, the problem is trivial. If x exceeds 1, however, gas dumps will have to be established at various points along the way. It will be convenient to take the path of the jeep along the positive x-axis, starting at x and ending at the origin. The gas dumps will then form a subdivision σ of the interval $(0, x)$:

$$\sigma: \quad 0 < x_1 < x_2 < \cdots < x_r < x,$$

in which the x_i denote the positions of the dumps (assumed to be finite in number). If z is any non-negative number less than x, the subdivision σ induces a subdivision of $(0, z)$ by deletion of all the stations to the right of z. There will be no ambiguity if we refer to this induced subdivision by the same symbol, σ. Other subdivisions will be denoted by σ', σ^*, and so forth. If all the stations (points of division) of σ are contained among the stations of σ', we shall say that σ' is a *refinement* of σ, written $\sigma' < \sigma$.

We may now rephrase our problem. Once a subdivision is fixed, the amount of gas required is still a function of the method of establishing and employing its stations. We shall denote by $f(x, \sigma)$ the greatest lower bound of this amount for all possible methods, and by $f(x)$ the greatest lower bound of $f(x, \sigma)$ for all possible subdivisions σ. Our task is to discover the form of $f(x)$.

In §2 we introduce the *standard* method of establishing and using the stations of a given subdivision σ, and we prove that this method is at least as economical as any other. This enables us to determine $f(x, \sigma)$ in §3. A rather surprising application of the standard method leads to the result (§4) that if $\sigma' < \sigma$, then $f(x, \sigma') \leqq f(x, \sigma)$. In §5 we determine criteria for non-improvement

by refinement. These criteria lead us to the construction of an optimum σ^* and to the explicit representation of $f(x, \sigma^*) = f(x)$ (§6). In §7 we derive a simple and accurate asymptotic formula for $f(x)$. The last section is devoted to a few remarks, including a comparison of the exact solution with the result obtained by considering the stations equally spaced (one of Niven's assumptions).

2. The standard method. One very natural method of employing the stations of a given σ is to build up the stockpile of gasoline at x_r by making all the trips between x and x_r before going to to x_{r-1}, and to continue in this way throughout the journey. In other words, once we go beyond any station x_i we never return to the preceding one, x_{i+1}.

Suppose that, by some other method, m complete round trips are made starting at x, followed by a last, one-way trip from x to x_r. The ith one of the round trips consists of A_i, the one-way trip from x to x_r; B_i, the round trip starting and ending at x_r; and C_i, the return trip from x_r to x. Let g_i be the amount of gas in the jeep at the start of A_i. Since $2(x-x_r)$ is the amount used in performing trips A_i and C_i, the amount $g_i-2(x-x_r)$ plus the residue of the preceding trips is sufficient to perform B_i. If we replace the sequence $A_1, B_1, C_1, A_2, B_2, C_2, \cdots, A_m, B_m, C_m, A_{m+1}$, by $A_1, C_1, A_2, C_2, \cdots, A_m, C_m, A_{m+1}$, and deposit at x_r the amount $g_i-2(x-x_r)$ after each $A_i (i=1, 2, \cdots, m)$, and $g_{m+1}-(x-x_r)$ after A_{m+1}, we shall then be in a position to perform all the B_i in exactly the same order as before. When this has been done, the final configuration will not have been altered and no more gas will have been used. The same reasoning applies to all the trips starting at x_r, and so, by induction, the standard method is established as being at least as economical as any other. Henceforth we shall assume its use.

3. Determination of $f(x, \sigma)$. Now we suppose that there is given a subdivision

$$\sigma: \quad 0 = x_0 < x_1 < \cdots < x_r < x_{r+1} = x,$$

and that $f(x_{i-1}, \sigma)$ has already been determined. Clearly $f(x_0, \sigma) = 0$, so we have the initial step in the inductive definition of $f(x_i, \sigma)$. Let k_i be the number of trips to be made from x_i to x_{i-1}. Obviously $k_i \geqq 1$. No gas is to be left behind at x_i, since that would imply waste. Hence the difference between $f(x_i, \sigma)$ and $f(x_{i-1}, \sigma)$ must be accounted for by the amount used in the $2k_i-1$ trips between the two stations. Writing $\Delta_i = x_i - x_{i-1}$, we have

$$f(x_i, \sigma) - f(x_{i-1}, \sigma) = (2k_i - 1)\Delta_i \qquad (i = 1, \cdots, r+1). \tag{1}$$

We must now determine k_i. The maximum amount of gas that can be transported on each of the first k_i-1 trips is $1-2\Delta_i$; on the last, $1-\Delta_i$, since there is no return. The total must not be less than the amount required to proceed from x_{i-1}. Hence

$$k_i(1 - 2\Delta_i) + \Delta_i \geqq f(x_{i-1}, \sigma). \tag{2}$$

From (1) it is clear that the number of trips must be as small as possible so that

$$(k_t - 1)(1 - 2\Delta_t) + \Delta_t < f(x_{t-1}, \sigma), \tag{3}$$

provided that $k_t > 1$. It is easy to see that $k_t = 1$ for $x_t \leqq 1$, and that $f(x_t, \sigma) = x_t$ in this case. If $x_t > 1$, we have $k_t > 1$. In this case, (2) and (3) determine the integer k_t uniquely, and $f(x_t, \sigma)$ is then obtained from (1). We remark that if the equality

$$m(1 - 2\Delta_t) + \Delta_t = f(x_{t-1}, \sigma)$$

holds for some integer m, then $k_t = m$.

We shall now derive a useful relationship between k_t and $f(x_t, \sigma)$. If we eliminate $f(x_{t-1}, \sigma)$ between (1) and (2), we have, for all $t \geqq 1$,

$$k_t \geqq f(x_t, \sigma). \tag{4}$$

Similarly, (1) and (3) yield

$$k_t - 1 < f(x_t, \sigma), \tag{5}$$

provided that $k_t > 1$. But we see directly that (5) is also valid for $k_t = 1$, so (4) and (5) hold for all $t \geqq 1$. If we define $\{a\}$ as the least integer not less than a, then for all $t \geqq 1$ we may write

$$k_t = \{f(x_t, \sigma)\}. \tag{6}$$

Since $f(z, \sigma)$ is an increasing function of z,

$$k_t \leqq k_{t+1}, \qquad t \geqq 1. \tag{7}$$

Summing (1), we obtain

$$f(x, \sigma) = \sum_{t=1}^{r+1} (2k_t - 1)\Delta_t. \tag{8}$$

4. Refinements of subdivisions. Let σ' be a refinement of σ. The quantity $f(x, \sigma)$ may be thought of as the result obtained by applying a *non-standard* method to σ', namely, passing over those stations of σ' which do not belong to σ. It follows immediately from §2 that $f(x, \sigma')$ is not greater than $f(x, \sigma)$, that is,

$$f(x, \sigma') \leqq f(x, \sigma) \qquad \text{if } \sigma' < \sigma. \tag{9}$$

If (x_{t-1}, x_t) is an interval of σ, with the associated parameter k_t, and if $(x_{t-1} = y_0, y_1), (y_1, y_2), \cdots, (y_{p-1}, y_p = x_t)$ are intervals of σ', with parameters $k', k'', \cdots, k^{(p)}$, then

$$k^{(p)} = \{f(x_t, \sigma')\} \leqq \{f(x_t, \sigma)\} = k_t. \tag{10}$$

Using (7), we obtain

$$k' \leqq k'' \leqq \cdots \leqq k^{(p)} \leqq k_t. \tag{11}$$

From (8),

$$f(x_t, \sigma') - f(x_{t-1}, \sigma') = \sum_{i=1}^{p} (2k^{(i)} - 1)(y_i - y_{i-1}). \tag{12}$$

Equation (1) may be written in the form

$$f(x_t, \sigma) - f(x_{t-1}, \sigma) = \sum_{i=1}^{p} (2k_i - 1)(y_i - y_{i-1}). \tag{13}$$

Subtracting the members of (12) from those of (13), we have

$$f(x_t, \sigma) - f(x_t, \sigma') = f(x_{t-1}, \sigma) - f(x_{t-1}, \sigma') + 2\sum_{i=1}^{p} (k_i - k^{(i)})(y_i - y_{i-1}). \tag{14}$$

We observe that all the differences in (14) are non-negative. From this we deduce that actual improvement by refinement takes place if and only if we can find an interval (x_{t-1}, x_t) and an integer i such that $k_t > k^{(i)}$. By (11), this is equivalent to $k_t > k'$.

5. Properties of σ^*. Our problem will be solved if we can find a subdivision σ^* for which

$$f(x, \sigma^*) \leqq f(x, \sigma) \quad \textit{for every } \sigma. \tag{A}$$

We can bring to bear the results of §4 by proving that any σ^* which satisfies (A) also satisfies (B) that follows, and conversely.

$$f(x, \sigma^*) = f(x, \sigma') \quad \textit{for every} \quad \sigma' < \sigma^*. \tag{B}$$

Clearly, (A) and (9) imply (B). Conversely, suppose that (B) is satisfied, and let σ be any subdivision whatsoever. We choose for σ' the common refinement of σ and σ^*. From (B), $f(x, \sigma^*) = f(x, \sigma')$; another application of (9) shows that $f(x, \sigma') \leqq f(x, \sigma)$. Combining these we obtain (A).

Using the criterion established at the end of §4, we find that (B) is equivalent to

(C) *For every $t = 1, 2, \cdots, r+1$, and for every y such that $x_{t-1} < y \leqq x_t$,*

$$k' \equiv \{f(y, \sigma^*)\} = k_t \equiv \{f(x_t, \sigma^*)\}.$$

We shall now show that (C) is equivalent to (D):

(D) *For every $m = 1, 2, \cdots, [f(x, \sigma^*)]$, there exists an integer s such that $f(x_s, \sigma^*) = m$.*

Suppose first that (C) fails for some t and y. Since $f(z, \sigma^*)$ is strictly increasing,

$$f(x_{t-1}, \sigma^*) < f(y, \sigma^*). \tag{15}$$

By the definitions of k' and the function $\{\ \}$,

$$f(y, \sigma^*) \leqq k'. \tag{16}$$

Since (C) fails, $k' < k_t$, that is,

$$k' \leqq k_t - 1. \tag{17}$$

Finally, since $k_t = \{f(x_t, \sigma^*)\}$ cannot exceed $f(x_t, \sigma^*)$ by as much as unity,

$$k_t - 1 < f(x_t, \sigma^*). \tag{18}$$

Combining (15), (16), (17), and (18), we see that the integer k' lies strictly between $f(x_{t-1}, \sigma^*)$ and $f(x_t, \sigma^*)$. By monotonicity, k' cannot be equal to $f(x_s, \sigma^*)$ for any s, so (D) fails.

Conversely, if m is an integer satisfying

$$f(x_{t-1}, \sigma^*) < m < f(x_t, \sigma^*), \tag{19}$$

we can find a number y such that

$$f(y, \sigma^*) = m, \tag{20}$$

and (C) is violated. To prove this, set

$$y = x_{t-1} + \frac{m - f(x_{t-1}, \sigma^*)}{2m - 1}, \tag{21}$$

and let $\Delta = y - x_{t-1}$. Clearly Δ is positive, and

$$m(1 - 2\Delta) + \Delta = f(x_{t-1}, \sigma^*). \tag{22}$$

Referring to the remark in §3, we see that m is the parameter associated with the interval (x_{t-1}, y), and

$$f(y, \sigma^*) = f(x_{t-1}, \sigma^*) + (2m - 1)(y - x_{t-1}) = m, \tag{23}$$

which proves (20). This completes the proof that (D) is equivalent to (C), (B), and (A).

6. Construction of σ^*. It is now almost trivial to construct a σ^* satisfying (D) and therefore (A). Merely choose the stations x_t^* so that $f(x_t^*, \sigma^*) = t$. Clearly this can be done for $t = 1$. Suppose that $x_1^*, \cdots, x_{t-1}^*$ have been found. We must determine x_t^* by

$$k_t = \{f(x_t^*, \sigma^*)\} = t, \tag{24}$$

$$f(x_t^*, \sigma^*) - f(x_{t-1}^*, \sigma^*) = (2k_t - 1)(x_t^* - x_{t-1}^*). \tag{25}$$

The left member of (25) equals unity; hence

$$x_t^* - x_{t-1}^* = (2t - 1)^{-1}. \tag{26}$$

Therefore

$$x_t^* = 1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2t - 1}. \tag{27}$$

It is easy to verify that (27) leads to the required equations $k_t = f(x_t^*, \sigma^*) = t$. The subivision σ^* so determined evidently satisfies (D). If r is the greatest integer for which x_r^* does not exceed x, we may write

$$f(x) = f(x, \sigma^*) = r + (2r + 1)(x - x_r^*), \tag{28}$$

and

$$0 \leqq x - x_r^* < (2r + 1)^{-1}. \tag{29}$$

It has now been shown that $f(x)$, which represents the number of gallons needed to take the jeep x miles, is a function which is piecewise linear over intervals of length 1, 1/3, 1/5, 1/7, and so on, the slope of the graph over the nth interval being the nth odd number so that the function takes consecutive integral values at the corner points.

7. An asymptotic formula for $f(x)$. From equation (1) it is possible to get a rough idea about the order of magnitude of $f(x)$. We have approximately

$$\Delta f = (2k - 1)\Delta x,$$
$$k = f(x).$$

Neglecting the -1 compared with k, we find

$$\frac{\Delta f}{f} = 2\Delta x,$$
$$\log f = 2x + C_1,$$
$$f = C_2 e^{2x}.$$

We shall not attempt to make these heuristic methods precise, but shall proceed directly to a derivation based on the exact solution obtained in the preceding section. Let us define

$$S(r) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{r}. \tag{30}$$

It is well known that for large r,

$$S(r) = \log r + C + \frac{1}{2r} + O\left(\frac{1}{r^2}\right), \tag{31}$$

where C is Euler's constant, .577 $\cdots$, and the $O(1/r^2)$ denotes an error term whose absolute value does not exceed a certain constant, independent of r, multiplied by $1/r^2$. This constant may vary from one equation to the next. From (27), (30), and (31), we have

$$x_r^* = S(2r) - \frac{1}{2}S(r) = \left(\log (2r) + C + \frac{1}{4r} + O\left(\frac{1}{r^2}\right)\right)$$
$$- \left(\frac{1}{2}\log r + \frac{1}{2}C + \frac{1}{4r} + O\left(\frac{1}{r^2}\right)\right). \tag{32}$$

Therefore,

$$x_r^* = \log(2\sqrt{r}) + \frac{1}{2}C + O\left(\frac{1}{r^2}\right).$$

Taking exponentials,

$$\exp\left(x_r^* - \frac{C}{2}\right) = 2\sqrt{r}\,\exp\left(O\left(\frac{1}{r^2}\right)\right) = 2\sqrt{r}\left(1 + O\left(\frac{1}{r^2}\right)\right). \tag{33}$$

Squaring, we obtain

$$\frac{1}{4}\exp(2x_r^* - C) = r + O\left(\frac{1}{r}\right). \tag{34}$$

Thus, writing $g(u) = \frac{1}{4}\exp(2u - C)$,

$$f(x_r^*) = r = g(x_r^*) + O\left(\frac{1}{r}\right) = g(x_r^*) + O(e^{-2x_r^*}). \tag{35}$$

Now for any u satisfying

$$x_r^* < u \leqq x < x_{r+1}^*, \tag{36}$$

we have, by equation (28),

$$f'(u) = 2r + 1. \tag{37}$$

Also,

$$g'(u) = 2g(u) > 2g(x_r^*) = 2r + O\left(\frac{1}{r}\right), \tag{38}$$

$$g'(u) < 2g(x_{r+1}^*) = 2r + 2 + O\left(\frac{1}{r}\right), \tag{39}$$

$$f'(u) - g'(u) = O(1). \tag{40}$$

Integrating (40) between x_r^* and x, we find

$$\begin{aligned} f(x) - g(x) &= (f(x_r^*) - g(x_r^*)) + \int_{x_r^*}^{x} (f'(u) - g'(u))\,du \\ &= O\left(\frac{1}{r}\right) + O(x - x_r^*) = O\left(\frac{1}{r}\right) \\ &= O(e^{-2x}). \end{aligned} \tag{41}$$

Therefore, for all x,

$$f(x) = \tfrac{1}{4}\exp(2x - C) + O(e^{-2x}). \tag{42}$$

It will be observed that the error term is not only of lower order than the principal term, but it actually tends to zero exponentially. Also, the approximate

equation derived by such high-handed methods at the beginning of this section turns out to be much more accurate than might have been expected.

8. Remarks. Suppose we decide to use equal subdivisions of the interval $(0, x)$, N in number. We can prove that if N is chosen to vary with x so that x^2/N tends to zero, then the increase in the amount of gas required (over the exact solution) is less than 50% for large x. To counterbalance this increase, however, the number of stations required will be very much smaller. For example, if N varies as the cube of x, it will also vary as the cube of $\log r$. Hence, if we take into account the cost (in time, energy, material, and so on) of setting up the stations, it might very well happen that the equal subdivisions would be more economical. Of course, we should then have an entirely new problem.

For a fixed x and sufficiently large N, we can come as close as we please to the minimum $f(x)$ by means of N equal subdivisions. An amusing fact here is that the minimum can be *attained* if and only if x is rational.

We close with several remarks about the character of the solution σ^* we have obtained here. It is obvious that this solution is not unique, since every refinement of σ^* is also a solution. It can be shown that the converse is also true; that is, every solution is a refinement of σ^*. Furthermore, if we consider the class of subdivisions σ for which the function $f(x, \sigma)$ is *continuous*, we can prove that this class is identical with the class of all solutions. It would be of interest to see whether this criterion can be obtained directly, and whether the minimum $f(x)$ can be derived from it.

MATHEMATICAL NOTES

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A LINEAR DIOPHANTINE EQUATION

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In solving the linear Diophantine equation

$$\sum_{i=1}^{n} a_i x_i = K, \qquad (a_1, a_2, \cdots, a_n) = 1,$$

where $(a_1, a_2 \cdots, a_n)$ denotes the greatest common divisor of $a_1, a_2, \cdots, a_n$, attention should be paid to the number of variables that may be considered arbitrary. Thus, if

$$(a_{i_1}, a_{i_2}, \cdots, a_{i_s}) = 1, \tag{1}$$

it is evident that $n-s$ of the variables may be assigned arbitrary values. The

maximum number of variables to which may be assigned arbitrary values is therefore $n-s$, where $s(\geqq 1)$ is the smallest number for which (1) holds, i_1, i_2, $\cdots$, i_s being any of the $C(n, s)$ choices of s subscripts from 1, 2, $\cdots$, n.

Suppose now that we have

$$a_1x_1 + a_2x_2 + \cdots + a_sx_s = K_1, \tag{2}$$

where $s>2$, and that

$$(a_1, a_2, \cdots, a_s) = 1,$$

while the latter condition does not hold for any proper sub-set of the a_i's. Subtracting a_ix_i from both members of (2), $1 \leqq i \leqq s$, we have

$$K_1 - a_ix_i = dt,$$

where

$$d = (a_1, a_2, \cdots, a_{i-1}, a_{i+1}, \cdots, a_s) > 1,$$

with obvious modification for $i=1$ or s; further, $(a_i, d) = 1$. Solving for x_i and t, we have

$$\frac{a_1}{d}x_1 + \frac{a_2}{d}x_2 + \cdots + \frac{a_{i-1}}{d}x_{i-1} + \frac{a_{i+1}}{d}x_{i+1} + \cdots + \frac{a_s}{d}x_s = t,$$

$$\left(\frac{a_1}{d}, \frac{a_2}{d}, \cdots, \frac{a_{i-1}}{d}, \frac{a_{i+1}}{d}, \cdots, \frac{a_s}{d}\right) = 1;$$

and the number of variables has been reduced by one. Continuing this process, we thus arrive at the general solution through successive solutions of the case involving two variables only. This may afford a rapid method of solution, especially if, at some step of the process, a coefficient 1 should appear or two coefficients should be relatively prime.

As an example, we consider the equation

$$99x + 79y + 55z + 33w = K,$$

discussed by D. H. Lehmer in this MONTHLY, vol. 48, 1941, p. 245. Since $(99, 79) = 1$, we may consider z and w arbitrary and write

$$99x + 79y = K - 55z - 33w. \tag{3}$$

Solving (3) by any of the well known methods, for example the Euclidean algorithmic process, we obtain

$$x = \quad 4K - 220z - 132w + 79t,$$
$$y = -5K + 275z + 165w - 99t,$$

with z, w, and t arbitrary. The reader may readily verify that this solution is equivalent to Lehmer's solution (21), p. 245, *loc. cit.*, namely

$$x = \quad 6K + 79L - 198M - 330N,$$
$$y = -5K - 66L + 165M + 275N$$
$$z = N,$$
$$w = -6K - 79L + 199M + 330N,$$

where L, M, and N are arbitrary; the equivalence is established by applying the linear transformation

$$L = 10K - 550z - 330w + 199t,$$
$$M = \quad 4K - 220z - 131w + \quad 79t,$$
$$N = z.$$

AN ARTILLERY PROBLEM

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A gun, shooting a heavy, low-velocity projectile, is trying to hit a target which need not lie on the same level as the gun. At what elevation should the gun fire, assuming a vacuum trajectory?

In an attempt to give a simple graphical method of determining the possible angles of elevation, the following construction was found.

Let the gun be at the origin of a rectangular coördinate system with horizontal x-axis and vertical y-axis, and let the target be at (x_0, y_0) in the xy-plane. Let

$$\xi_0 = \frac{gx_0}{V^2}, \qquad \eta_0 = \frac{gy_0}{V^2},$$

where V is the muzzle velocity of the projectile and g is the acceleration due to gravity. On the figure we carry out the following construction:

1. Through T: (ξ_0, η_0) draw a line parallel to the y-axis, meeting the hyperbola $xy=1$ in A.
2. Through A draw a line AC parallel to the x-axis.
3. Through the origin O draw OC perpendicular to OT.
4. With C, the intersection of AC and OC, as center, draw a circle through O, cutting the line $x=1$ in R_1 and R_2.
5. Draw OR_1 and OR_2, making angles of α_1 and α_2 with the positive x-axis.

Now α_1 and α_2 are the required angles of elevation.

The proof of this construction is simple. In terms of the normalized distances $\xi = gx/V^2$ and $\eta = gy/V^2$ and the normalized time $\tau = gt/V$, the equations of motion of the projectile are

$$\xi = \tau \cos \alpha, \qquad \eta = \tau \sin \alpha - \tfrac{1}{2}\tau^2.$$

Eliminating τ gives

$$\eta = m\xi - \tfrac{1}{2}(1 + m^2)\xi^2,$$

where $m = \tan \alpha$. We wish m to satisfy

$$\eta_0 = m\xi_0 - \tfrac{1}{2}(1 + m^2)\xi_0^2. \tag{1}$$

Now the circle

$$\eta_0 x = y\xi_0 - \tfrac{1}{2}(x^2 + y^2)\xi_0^2 \tag{2}$$

intersects the line $x = 1$ in points whose ordinates are roots of (1), and so the lines joining these points to O have for slopes the required values of m. It is readily verified that C is the center of (2).

Additional information can be obtained from the figure that follows:

1. The normalized time of flight is $\tau_0 = \xi_0 \sec \alpha$. This is just the distance from O to the point in which OR intersects TA.

2. The parabola YZ, with focus at O, determines the maximum range of the projectile. T must be inside this parabola.

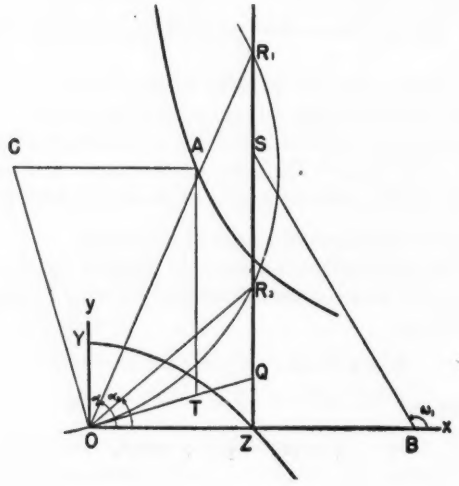
3. The angle of fall ω of the projectile at T is given by

$$\tan \omega = \left(\frac{d\eta}{d\xi}\right)_T = m - (1 + m^2)\xi_0.$$

Substituting for $1 + m^2$ from (1) gives

$$\tan \omega = \frac{2\eta_0}{\xi_0} - m.$$

Now ω is readily constructed by extending OT to meet the line $x = 1$ in Q and marking off $RS = 2QZ$ downward from R. Then ω is the angle between the positive x-axis and BS, where B is the point (2, 0).



CLASSROOM NOTES

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All material for this department should be sent to C. B. Allendoerfer, Haverford College, Haverford, Pennsylvania. Contributions are invited on topics of immediate interest to teachers of undergraduate mathematics such as: fresh approaches to standard material, analyses of common textbook shortcomings, descriptions of visual and mechanical aids to teaching, outlines of new types of courses, and discussions of the role of mathematics in the revised curricula being adopted by many institutions. Rejoinders to earlier notes are encouraged.

A RIGOROUS TREATMENT OF THE FIRST MAXIMUM PROBLEM IN THE CALCULUS

J. L. WALSH, Harvard University

Even some of the best calculus texts fail to set forth adequately the logic underlying the usual maximum and minimum problems, studied subsequently to curve-plotting. The present note outlines the treatment the writer has used for a number of years, in a course primarily for freshman and sophomores.

PROBLEM. *A piece of sheet tin three feet square is to be made into a rectangular box open at the top, by cutting out equal squares from the corners and bending up the sides of the resulting piece parallel with the edges. Among all such boxes, to find the box of greatest volume.*

We consider an arbitrary box that can be made in this way. Let x denote the length in feet of the side of the square corners cut out, so that x is also the height of the box. The volume V of the box is then (in cubic feet)

$$V = V(x) = x(3 - 2x)^2. \tag{1}$$

We study V as a function of x in the *closed* interval $0 \leqq x \leqq 3/2$. When x is very small and positive, so also is the volume V, as appears both by (1) and by considering the shallow box formed; we have $V(0) = 0$. When x is smaller than but near $3/2$, the volume V is also very small and positive as is seen both by (1) and by considering the small area of the base of the corresponding box, we have $V(3/2) = 0$. Between these two values $x = 0$ and $x = 3/2$, the continuous function V is always positive and hence possesses a maximum; the maximum V is characterized by the vanishing of $D_x V$ for the corresponding value of x. We find

$$D_x V = x \cdot 2(3 - 2x)(-2) + (3 - 2x)^2 = (3 - 2x)(3 - 6x).$$

Between the two values 0 and $3/2$ the only value of x for which $D_x V$ vanishes is $x = 1/2$, which therefore yields the box of maximum volume.

The logic underlying this formal work can be analyzed as follows:

1. The function $V(x)$ is continuous in the closed interval $0 \leqq x \leqq 3/2$, hence possesses a maximum* there. [Any function continuous in a closed finite interval possesses a maximum there.]

2. The function V vanishes for $x = 0$ and $x = 3/2$ and is positive between

* To be more explicit, $V(x)$ possesses a (weak) absolute maximum, which is necessarily also a (weak) relative maximum.

those values, so the maximum occurs at some point or points in the *interior* $0<x<3/2$ of the original interval.

3. At this maximum of $V(x)$ we must have $D_xV=0$. [If $D_xV>0$, the function V increases as x increases; if $D_xV<0$, the function V decreases as x increases and increases as x decreases; in either case V has no maximum.]

4. Among the values $0<x<3/2$ we have $D_xV=0$ if and only if $x=1/2$.

5. Hence $x=1/2$ yields the maximum value of V.

It will be noticed that $x=1/2$ cannot yield a minimum of V, nor a point of inflection. Moreover, the function $V(x)$ possesses a *unique* maximum in the interval $0\leqq x\leqq 3/2$; there is no choice here among several relative maxima.

The logic as given is complete (although the theorem quoted in 1 can hardly be proved at this stage), and will satisfy the most critical student even at any stage of his later career. This logical analysis, with suitable modifications, applies to all maximum-minimum problems in one variable that the student is likely to meet.

TEACHING TRIGONOMETRY

E. P. VANCE, Oberlin College

A frequent source of confusion in mathematics courses in college is the meaning of $\sin x$, $\cos x$, and so forth, when x is a general real variable. In dealing with periodic behavior or phenomena resulting from combinations of periodic behavior, such as in the study of electricity, one frequently encounters expressions such as $\sin \omega t$. Because of the manner in which the circular functions were originally defined, often the student will ask: "Where is the angle?" Beginners are in the habit of thinking of these functions solely as functions of angles and many are incapable of understanding any generalization. The author feels that steps should be taken to clarify this difficulty in the students' minds.

On an experimental basis we are attacking this problem in some classes at Oberlin by defining the functions at the outset as functions of a real variable θ. The method consists of drawing a unit circle with its center at the origin of a rectangular axis system. Then corresponding to the real variable θ, an arc of length θ is laid off from (1, 0) along the circle; if θ is positive the arc is laid off counterclockwise, but if θ is negative the direction of the arc is clockwise. Letting P be the endpoint of this arc, the x-coördinate of P is defined to be $\cos\theta$, and the y-coördinate of P is defined to be $\sin\theta$. The other trigonometric functions may then be defined in terms of the ratios and reciprocals of the sine and cosine functions. It will be immediately apparent that all the properties of the circular functions easily follow: $\cos 0=1$, $\sin 0=0$, $\sin(-\theta)=-\sin\theta$, $\sin(\pi/2-\theta)=\cos\theta$, $\sin^2\theta+\cos^2\theta=1$, and so forth. Problems in analytic trigonometry, such as identities and equations, are always phrased in terms of the real variable θ. The graphs and the use of graphs play a dominate role, thus emphasizing the important periodic properties of these functions. In constructing the graph of $y=\sin x$, care is taken to use the same type of units for both x and y, although not necessarily the same scale. It is then pointed out that the radian measure of

an angle is equal to the value of θ of the intercepted arc, and thus the functions of an angle are derived as special cases. The angle approach is used only for the solutions of triangles.

An alternative approach is to use standard textbooks up to the point where graphs are introduced. Then the generalization may be made as described above. Care, however, should be taken to bridge the gap between the notions of sin (2 radians) and sin 2; for in the former, angle is still predominant while in the latter the full transfer to real variables has been carried out.

The sophisticate may object that in this approach, trigonometry is based upon the notion of arc length, generally agreed to be a more advanced mathematical topic. In fact, G. H. Hardy refuses to discuss arc length even in a book as advanced as his *Pure Mathematics* because of its complexity. The reply is that to freshmen arc length is a perfectly clear concept, and its use is very helpful in the understanding of this matter. A rigorous approach in terms of integrals leading to inverse functions may well find its place in the calculus.

UNDETERMINED COEFFICIENTS IN INTEGRATION

J. B. Reynolds, Lehigh University

Teachers of calculus usually find difficulty in getting students to be able to integrate readily expressions in which the integrand is a product of an exponential term and a binomial factor each term of which contains a sine or a cosine of an angle. The integration is usually accomplished by integrating by parts twice. Sometimes a table of integrals is resorted to and the student never really understands how the integral was found. The integration of the forms cited can be performed readily by the use of undetermined coefficients. Suppose we want to integrate the following product

$$e^{3x}(2 \sin 5x - 4 \cos 5x)dx.$$

A little thought shows that the only terms the integral can contain are given in

$$\int e^{3x}(2 \sin 5x - 4 \cos 5x)dx = e^{3x}(A \sin 5x + B \cos 5x) + C$$

in which A and B are coefficients to be determined. Upon taking the derivative of both sides with respect to x we have

$$e^{3x}(2 \sin 5x - 4 \cos 5x) = e^{3x}([3A - 5B] \sin 5x + [3B + 5A] \cos 5x).$$

Upon cancelling e^{3x} from both sides and equating coefficients of $\sin 5x$ and $\cos 5x$ we have

$$3A - 5B = 2 \quad \text{and} \quad 3B + 5A = -4$$

whence $A=-7/17$ and $B=-11/17$ and, therefore,

$$\int e^{3x}(2\sin 5x - 4\cos 5x)dx = -e^{3x}(7\sin 5x + 11\cos 5x)/17 + C.$$

It is equally easy to apply this method to the integral of the product of a sine and a cosine of two angles. Thus, if we asume

$$\int \sin 3x \cos x\,dx = A\sin 3x \sin x + B\cos 3x \cos x + C$$

we find after differentiating and equating coefficients of like terms $A-3B=1$ and $3A-B=0$, giving $A=-1/8$ and $B=-3/8$.

Of course the method can be used to derive general formulas for the integration of these and other forms.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, Oregon State College

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Oregon State College, Corvallis, Oregon. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 751. *Proposed by Alan Wayne, Flushing, N. Y.*

Find the digits represented by the letters in the following addition, if no two different letters represent the same digit:

$$\begin{array}{r} F\ O\ R\ T\ Y \\ T\ E\ N \\ T\ E\ N \\ \hline S\ I\ X\ T\ Y \end{array}$$

E 752. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Show that in a right triangle the twelve points of contact of the inscribed and escribed circles form two groups of six points situated on two circles which cut each other orthogonally at the points of intersection of the circumcircle with the line joining the midpoints of the legs of the triangle.

E 753. *Proposed by L. M. Kelly, University of Missouri*

How can one convince a class in elementary analytics that if the inside of a

race track is a non-circular ellipse, and the track is of constant width, then the outside is not an ellipse?

E 754. *Proposed by C. A. Richmond, Tyngsboro, Mass.*

A finite sequence of positive integers will be said to be *skew ordered* if either each integer in an even position of the sequence is greater than or each such integer is less than its immediate neighbors. If the eight integers $1, \cdots, 8$ are placed in random order in a sequence, what is the probability that the sequence will be skew ordered?

E 755. *Proposed by Alfred Brauer, University of North Carolina*

Let a_1, a_2, a_3, a_4 be relatively prime integers such that

$$a_1^3 + a_2^3 + a_3^3 + a_4^3 = 0. \tag{1}$$

Let $\alpha_\nu (\nu = 1, 2, 3, 4)$ be the smallest non-negative residue of $a_\nu \pmod 6$. Then

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \equiv 0 \pmod 6. \tag{2}$$

In his paper, *Residual Types of Partitions of* 0 *into Four Cubes* (The Mathematics Student, vol. 13, 1945, pp. 47–48), A. K. Srinivasan tries to find solutions of (1) for each set of numbers α_ν satisfying (2). For instance, for $\alpha_1 = \alpha_2 = 0$, $\alpha_3 = 1$, $\alpha_4 = 5$, he gives the solution $a_1 = 12$, $a_2 = -54$, $a_3 = 19$, $a_4 = 53$. In the following cases he did not succeed in finding examples:

$$\begin{cases} \alpha_1 = 0, \quad \alpha_2 = \alpha_3 = 1, \quad \alpha_4 = 4; \\ \alpha_1 = \alpha_2 = \alpha_3 = 1, \quad \alpha_4 = 3; \\ \alpha_1 = \alpha_2 = 2, \quad \alpha_3 = 3, \quad \alpha_4 = 5; \end{cases} \tag{3}$$

and in the cases obtained from (3) if each α_ν is replaced by $6 - \alpha_\nu$. He conjectures that these cases are impossible. Prove that this conjecture is true.

SOLUTIONS

A Four Digit Square

E 722 [1946, 270]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Find a square of four digits such that if one interchanges the two middle digits the new number is a square for radix 9.

Solution by Walter Penney, Washington, D. C. Let the four digits, in order, be a, b, c, d. The only final digits for $(\text{squares})_{10}$ are 0, 1, 4, 5, 6, and 9, and for $(\text{squares})_9$ are 0, 1, 4, and 7. Therefore d must be 0, 1, or 4.

If $d = 0$, c must be 0, since all $(\text{squares})_{10}$ ending in 0 end in 00. All $(\text{squares})_9$ ending in 0 end in 10, 40, or 70. Since no $(\text{square})_{10}$ can end in 700, b must be 1 or 4. The only possibility is $(8100)_{10}$. But $(8010)_9$ is not a square. Therefore $d \neq 0$.

Similarly, considering the penultimate digit when $d = 1$, we find that no $(\text{square})_{10}$ is also a $(\text{square})_9$ when b and c are interchanged. Therefore $d \neq 1$.

If $d=4$, we find

$$(3364)_{10} = 58^2 \quad \text{and} \quad (3634)_9 = 57^2,$$
$$(4624)_{10} = 68^2 \quad \text{and} \quad (4264)_9 = 62^2.$$

These are therefore the only two solutions.

Also solved by W. E. Byrne, Monte Dernham, and E. P. Starke. Margaret Olmsted found the solution 3364; Paul Bateman, C. Perry, and the proposer found the solution 4624. Starke pointed out that tables of squares in other systems than the decimal are to be found in the proposer's pamphlet, *Les Recréations Mathématiques* (parmi les nombres curieux), Supplément a Mathesis, 1943. With these in hand the answers can be read off directly.

Trigonometric Inequalities

E 723 [1946, 271]. *Proposed by A. W. Goodman, Columbia University*

Let A and B be any real quantities, and let k be a positive integer. Show that

$$\left|\frac{\cos kB \cos A - \cos kA \cos B}{\cos B - \cos A}\right| \leqq k^2 - 1, \tag{1}$$

$$\left|\frac{\cos kB - \cos kA}{\cos B - \cos A}\right| \leqq k^2, \tag{2}$$

equality holding only in the limit as B and A approach zero.

I. *Solution by A. S. Peters, New York University.* The fundamental inequality upon which the proposed inequalities depend is

$$|\sin rx| \leqq r|\sin x|, \tag{3}$$

where r is a positive integer. This inequality is easily proved, for

$$\begin{aligned} |\sin rx| &= |\sin [(r-1)x + x]| \\ &= |\sin (r-1)x \cos x + \cos (r-1)x \sin x| \\ &\leqq |\sin (r-1)x| + |\sin x|, \end{aligned}$$

from which (3) follows by iteration. Now, from (3),

$$\left|\frac{\sin rx \sin sy}{\sin x \sin y}\right| \leqq rs,$$

or, more symmetrically

$$\frac{1}{2}\left|\frac{\sin rx \sin sy + \sin sx \sin ry}{\sin x \sin y}\right| \leqq rs, \tag{4}$$

where r and s are positive integers and the equality sign holds only if $r=s=1$; or $x \to 0$ and $y \to 0$. With the substitutions $r=m+n$, $s=m-n$, $x=(A+B)/2$,

$y=(A-B)/2$, and the use of some trigonometric identities, (4) becomes

$$\left|\frac{\cos mB\cos nA-\cos mA\cos nB}{\cos B-\cos A}\right|\leqq m^2-n^2, \tag{5}$$

where $m>n$ and $m=(r+s)/2$, $n=(r-s)/2$. The inequalities (1) and (2) are particular cases of (5). If $m=k$, $n=1$, then (5) reduces to (1). If $m=k$, $n=0$, then (5) reduces to (2). Other inequalities of the same nature as (5) can be deduced by analyses similar to the above.

II. *Solution by R. C. Buck, Harvard University.* Let $T_n(t)$ be the Tchebycheff polynomial of degree n; this has the value $\cos n\theta$ if $t=\cos\theta$. Let P and Q be the points on the curve $y=T_k(x)$ with x-coordinates $\cos B$ and $\cos A$, and let L be the line through them. Then, the left member of (1) is the absolute value of the y-intercept of L, and the left member of (2) is the absolute value of the slope of L. Since L is a secant of the curve, its slope cannot exceed the greatest value of $|T_k'(t)|$, which is known to be k^2, thus proving (2). Since the curve lies entirely within the 2-unit square, the greatest value of the y-intercept of L is obtained by finding the greatest value of the y-intercept of a tangent line for $|t|\leqq 1$. But, for tangents, the intercept is given by $|T_k(t)-tT_k'(t)|$, whose greatest value is k^2-1, thus proving (1).

The facts stated above for the Tchebycheff polynomials are most easily seen from the relations

$$T_n'(t)=n\sin n\theta/\sin\theta,$$

$$tT_n'(t)-T_n(t)=n\sin(n-1)\theta/\sin\theta+(n-1)\cos n\theta.$$

Thus

$$|T_n'(t)|\leqq n\left|\sin n\theta/\sin\theta\right|\leqq n^2,$$

while

$$|T_n(t)-tT_n'(t)|\leqq n(n-1)+n-1=n^2-1.$$

Equality in (1) and (2) holds as A and B approach 0 or π.

Also solved by Paul Brock and Alan Wayne.

Admissible Numbers

E 724 [1946, 271]. *Proposed by N. J. Fine, University of Pennsylvania, and Ivan Niven, Purdue University*

Define an n-admissible number k as one such that an n-dimensional cube may be subdivided into k cubes. Prove that for each n there exists an integer A_n such that all integers exceeding A_n are n-admissible.

Solution by Fritz Herzog, Michigan State College. Let k be an n-admissible number, and let an n-dimensional cube be subdivided into k cubes. If one of these k cubes is further subdivided into a^n equal cubes, where a is any integer

greater than unity, then we obtain a subdivision of the original cube into $k+(a^n-1)$ cubes. Thus, if k is n-admissible, so is $k+(a^n-1)$. Since unity is n-admissible, we conclude that all numbers of the form

$$1 + p(a^n - 1) + q(b^n - 1), \qquad a \geq 2, b \geq 2, p \geq 0, q \geq 0, \tag{1}$$

are n-admissible. Choose a and b so that $(a^n-1, b^n-1)=1$, which is always possible, for instance by taking $a=2$, $b=2^n-1$. Then, by a well known fact in number theory (see, *e.g.*, Polya-Szegö, *Aufgaben und Lehrsätze aus der Analysis*, vol. 1, p. 4, Problem 26), every integer $k>(a^n-1)(b^n-1)$ can be written in the form (1) in at least one way, and is, therefore, n-admissible.

Also solved by Paul Bateman, J. B. Kelly, Leo Moser, William Scott, and the proposers.

Editorial Note. The problem of finding the largest number not n-admissible appears not to be easy. For $n=2$ the answer is 5. Scott showed this by proving that the only numbers not 2-admissible are 2, 3, and 5. Following Herzog's suggestion of taking $a=2$, $b=2^n-1$, we find that all numbers greater than 2394 are 3-admissible. This was greatly improved upon by Scott, who showed that 1, 20, 38, 39, 49, 51, 61 are all 3-admissible. These numbers, along with those obtained by adding multiples of 7 to these, then constitute a set of 3-admissible numbers, and it is easily shown that every number greater than 54 is in this set. The question as to whether 54 is 3-admissible or not is still open.

Suppose we define a number k to be *strictly* n-admissible if an n-dimensional cube may be subdivided into k *different* cubes. For some time the conjecture was that there are no numbers strictly 2-admissible. This conjecture was proved false in the paper *The Dissection of Rectangles into Squares*, by C. A. B. Smith, A. H. Stone, W. T. Tutte, Duke Mathematics Journal, Dec. 1940. This same paper proved that there are no numbers which are strictly 3-admissible.

Bateman remarked that the proposed problem appeared in Ripley's "Believe It or Not" column.

Correlation Coefficient Between Two Events

E 725 [1946, 271]. *Proposed by Henry Scheffé, University of California at Los Angeles*

If two events have probabilities p_1 and $p_2(0<p_i<1)$, and correlation coefficient ρ, show that the range of possible ρ is the interval $-f(a_1a_2) \leqq \rho \leqq f(a_1/a_2)$, where $a_i=[p_i/(1-p_i)]^{1/2}$ and $f(x)=\min\ (x,\ 1/x)$. The correlation coefficient between two events E_1 and E_2 may be defined as that between x_1 and x_2, where $x_i=1$ if E_i happens and $x_i=0$ if not-E_i happens.

Solution by Ellen Buck, Wellesley College. By definition

$$\rho = \frac{M(x_1x_2) - M(x_1)M(x_2)}{[M(x_1^2) - (M(x_1))^2]^{1/2}[M(x_2^2) - (M(x_2))^2]^{1/2}}$$

where $M(x_i)$ denotes the mean value of x_i. Now, if p is the probability that both E_1 and E_2 happen, we have

$$M(x_i) = p_i, \qquad M(x_i^2) = p_i, \qquad M(x_1x_2) = p,$$
$$\max(p_1 + p_2 - 1, 0) \leqq p \leqq \min(p_1, p_2).$$

Therefore

$$\frac{\max(p_1 + p_2 - 1, 0) - p_1p_2}{p_1(1 - p_1)p_2(1 - p_2)} \leqq \rho \leqq \frac{\min(p_1, p_2) - p_1p_2}{p_1(1 - p_1)p_2(1 - p_2)}.$$

The numerators may be simplified to

$$\max[-(1 - p_1)(1 - p_2), -p_1p_2], \qquad \min[p_1(1 - p_2), p_2(1 - p_1)]$$

respectively. Then, in terms of the a_i, our inequality becomes

$$-\min(a_1a_2, 1/a_1a_2) \leqq \rho \leqq \min(a_1/a_2, a_2/a_1),$$

which is the desired relation.

Also solved by Paul Bateman.

Professor Umbugio's Prediction

E 716 [1946, 219], *Proposed by H. E. G. P.*

On April 1, 1946, the Erehwon Daily Howler carried the following item: "The famous astrologer and numerologist of Guayazuela, the Professor Euclide Paracelso Bombast Umbugio, predicts the end of the world for the year 2141. His prediction is based on profound mathematical and historical investigations. Professor Umbugio computed the value of the formula

$$1492^n - 1770^n - 1863^n + 2141^n$$

for $n = 0, 1, 2, 3$, and so on, up to 1945, and found that all the numbers which he so obtained in many months of laborious computation are divisible by 1946. Now, the numbers 1492, 1770, and 1863 represent memorable dates: the Discovery of the New World, the Boston Massacre, and the Gettysburg Address. What important date may 2141 be? That of the end of the world, obviously."

Deflate the Professor! That is, show with little computation that the formula proposed is divisible by 1946 for $n = 0, 1, 2, 3, \cdots$.

Solution by E. P. Starke, Rutgers University. That $x - y$ is a divisor of $x^n - y^n$ for $n = 0, 1, 2, \cdots$, is the only principle required. Let the Professor's number be $F(n)$. Then, since $2141 - 1863 = 1770 - 1492 = 278$, we certainly have $F(n)$ is always divisible by 278. Similarly, since $2141 - 1770 = 1863 - 1492 = 371$, $F(n)$ is always divisible by 371. But 278 and 371 are relatively prime, whence $F(n)$ is always divisible by $(278)(371) = (53)(1946)$, and hence, of course, by 1946 itself.

Also solved by D. W. Alling, Murray Barbour, Joshua Barlaz, Paul Bateman, Barney Bissinger, W. G. Brady, Paul Brock, D. H. Browne, R. C. Buck, Roy Dubisch, George Grossman, E. S. Keeping, W. J. LeVeque, C. L. Peny, Jr., P. A. Pizá, J. G. Wendel, Maud Willey, R. H. Wilson, Jr., and the proposer.

Buck pointed out that more generally

$$A^n + (A + r + s)^n - (A + r)^n - (A + s)^n$$

is always divisible by the least common multiple of r and s.

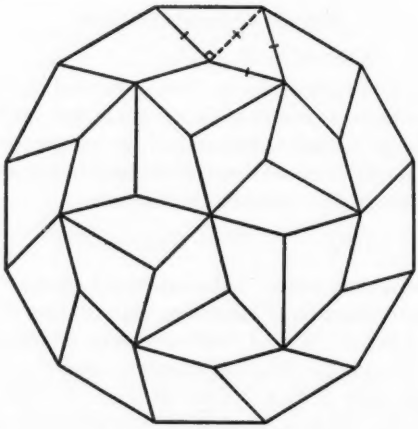
Pizá remarked that the extra factor 53 may have significance, because that is the number of cards in a pack when the joker is included, and surely the proposer of the problem must have been the joker.

A Dodecagon Dissection Puzzle

E 721 [1946, 270]. *Proposed by Joseph Rosenbaum, The Milford School, Conn.*

Given two equal regular dodecagons. Show how to dissect one of them into twelve congruent pieces which can be fitted to the other to form another larger regular dodecagon.

Solution by J. M. Kingston, University of Washington. The dissection is apparent from the following figure. Note that six of the quadrilaterals must be turned over after dissection.



Also solved, in the same way, by E. S. Smith and the proposer.

Editorial Note. For other dissection puzzles see Dudney, *Amusements in Mathematics*, and Ball-Coxeter, *Mathematical Recreations and Essays*. The latter gives further references.

Matching Chips

E 717 [1946, 219]. *Proposed by Orrin Frink, Jr., State College, Pa.*

In the game called "matching chips" each player selects, independently of the other, either a red or a white chip from his store of chips. If both chips are found to be of the same color, then player A keeps both chips; otherwise player B keeps both chips. Red chips are worth r cents each, and white chips are worth

w cents each. Develop the theory of this game.

Solution by the Proposer. If A selects red chips with probability (or frequency) p, and B selects red chips with probability q, then A's mathematical expectation M is: $M=rpq+w(1-p)(1-q)-rp(1-q)-wq(1-p)$. Hence $M=(2q-1)(rp+wp-w)$.

It follows that A's "best strategy" is to select red chips with probability $w/(r+w)$, and B's "best strategy" is to select red and white chips with equal probability $\frac{1}{2}$. If *either* player adopts his best strategy, the mathematical expectation of *both* players is zero. Hence the game is "fair."

If B departs from his best strategy, then A can take advantage of this error by imitating it; while if A departs from his best strategy, B can penalize him by reversing the error.

A general theory of games of this sort is given in *Theory of Games and Economic Behavior* by von Neumann and Morgenstern, Princeton, N. J., 1944.

Derangements

E 719 [1946, 220]. *Proposed by George Grossman, New York City*

A man addresses n envelopes and writes n checks in payment of n bills.

(a) If the n bills are placed at random in the n envelopes, what is the probability that each bill is placed in a wrong envelope?

(b) If the n bills and the n checks are placed at random in the n envelopes, one of each in each envelope, what is the probability that in no instance are the enclosures completely correct?

(c) In (b), what is the probability that each bill and each check will be in a wrong envelope?

Solution by Paul Bateman, Philadelphia, Pa. We use the well known combinatorial theorem: If there are N objects, of which $N(A_1)$ have the property A_1, $N(A_2)$ have the property A_2, $\cdots$, $N(A_1A_2)$ have both A_1 and A_2, $\cdots$, $N(A_1A_2A_3)$ have A_1, A_2, and A_3, $\cdots$, and so on, then the number of objects which have none of A_1, A_2, A_3, $\cdots$ is

$$N - N(A_1) - N(A_2) - \cdots + N(A_1A_2) + \cdots - N(A_1A_2A_3) - \cdots .$$

Part (a) is the classical "problème des rencontres." The total number of ways of putting n bills in n envelopes is $n!$. If these $n!$ ways are regarded as the objects of the preceding theorem and a way is considered to have property A_1 if under it the first envelope has the right bill in it, to have property A_2 if the second envelope has the right bill in it, *etc.*, then we find for the number of ways such that no envelope has the right bill:

$$n! - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! - \cdots + (-1)^n\binom{n}{n} = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!}\right).$$

Hence the desired probability is

$$\sum_{i=2}^{n} (-1)^i/i!.$$

A similar application of the combinatorial theorem to the $(n!)^2$ ways of putting n bills and n checks in n envelopes gives the following as the number of ways such that in no envelope are both enclosures correct:

$$(n!)^2 - \binom{n}{1}\{(n-1)!\}^2 + \binom{n}{2}\{(n-2)!\}^2 - \cdots + (-1)^n\binom{n}{n}$$
$$= (n!)^2\left\{1 - \frac{1}{1!n} + \frac{1}{2!n(n-1)} - \cdots + \frac{(-1)^n}{n!n(n-1)\cdots 1}\right\}.$$

Hence the desired probability for part (b) is

$$\sum_{i=0}^{n} (-1)^i(n-i)!/i!n!.$$

The probability asked for in part (c) is merely the square of that obtained for part (a).

A somewhat more difficult problem is that of finding the probability that each bill and each check will be in a wrong envelope and no bill will be in the same envelope with the check which is supposed to go with it.

Also solved by D. W. Alling, Murray Barbour, D. H. Browne, Harley Flanders, and E. P. Starke.

Browne pointed out that the distribution in part (b) is ${}_nH_n$ of John Riordan's solution to *E* 589 [1944, 287]. He also showed that with increasing n the three probabilities of parts (a), (b), and (c) respectively approach e^{-1}, 1, e^{-2}.

Editorial Note. For a proof (and several interesting applications) of the combinatorial theorem referred to see, *e.g.*, Chapter V of *Elementary Number Theory* by Uspensky and Heaslet. Also, in connection with this problem, see problem 4146 and editorial note [1946, 107].

The Extended Coin Problem

E 712 [1946, 156]. *Proposed by Donald Eves, Paterson, N. J.*

A man has twelve coins, all of which appear exactly alike, but one of which is counterfeit and does not weigh the same as a genuine coin. He has at his disposal a delicate set of balances, but no weights. How can he detect the false coin, and whether light or heavy, in not more than three weighings? (*Cf.* E 651 [1945, 397].)

I. *Solution by E. D. Schell, Arlington, Va.* Label the coins 1, 2, $\cdots$, 12. When an equal number of coins are placed on each side of the scale, the left side may be heavier, the coins may balance, or the right side may be heavier. Denote these three situations by *L*, *B*, and *R* respectively. Further, let

$(a, b, c, \cdots)-(x, y, z, \cdots)$ mean the coins $a, b, c, \cdots$ are weighed on the left against the coins $x, y, z, \cdots$ on the right. $H(x)$ will mean that coin x is counterfeit and heavier than the others, while $S(x)$ will mean coin x is counterfeit and short in weight. A method of weighing which leads to a solution is as follows:

$$
(1, 2, 3, 4)-(5, 6, 7, 8)\begin{cases}
L\ (1, 2, 6)-(3, 4, 5)\begin{cases}
L\ (1)-(2)\begin{cases} L\ H(1) \\ B\ S(5) \\ R\ H(2) \end{cases} \\
B\ (7)-(8)\begin{cases} L\ S(8) \\ R\ S(7) \end{cases} \\
R\ (3)-(4)\begin{cases} L\ H(3) \\ B\ S(6) \\ R\ H(4) \end{cases}
\end{cases} \\
B\ (1, 2, 3)-(10, 11, 12)\begin{cases}
L\ (10)-(11)\begin{cases} L\ S(11) \\ B\ S(12) \\ R\ S(10) \end{cases} \\
B\ (1)-(9)\begin{cases} L\ S(9) \\ R\ H(9) \end{cases} \\
R\ (10)-(11)\begin{cases} L\ H(10) \\ B\ H(12) \\ R\ H(11) \end{cases}
\end{cases} \\
R\ (1, 2, 6)-(3, 4, 5)\begin{cases}
L\ (1)-(2)\begin{cases} L\ S(2) \\ B\ H(5) \\ R\ S(1) \end{cases} \\
B\ (7)-(8)\begin{cases} L\ H(7) \\ R\ H(8) \end{cases} \\
R\ (3)-(4)\begin{cases} L\ S(4) \\ B\ H(6) \\ R\ S(3) \end{cases}
\end{cases}
\end{cases}
$$

It is interesting to note that it is sufficient to be given twelve coins in order to determine the existence of the counterfeit among them and its identity. That is, the existence of the counterfeit coin is not used in solving the problem and is not essential to the hypothesis.

In general we may determine the existence of a counterfeit and its identity among $(3^n-1)/2$ coins in n weighings if we are given an additional 3^{n-1} coins known not to be counterfeit. From this result it is easy to show that the exist-

ence and identity of a counterfeit coin may be determined in n weighings if $(3^n-3)/2$ coins are given without any additional coins.

II. *Solution by Joseph Rosenbaum, The Milford School, Conn.* Label the coins, 1, 2, $\cdots$, 12, and make the following three weighings:

(1) 1, 2, 3, 4 against 5, 6, 7, 8,

(2) 1, 2, 3, 5 against 4, 9, 10, 11,

(3) 1, 6, 9, 12 against 2, 5, 7, 10.

It will now be shown how the counterfeit coin can be detected, and whether light or heavy, from the results of these three weighings. Designate the relationships "is lighter than," "is the same as," "is heavier than" by L, S, H, and denote by x, y, z, respectively, the observed relationships of the left to the right in (1), (2), (3). It is now easy to verify that out of the $27=3^3$ permutations with repetitions of L, S, H the three $(x, y, z)=(S, S, S)$, (L, H, H), and (H, L, L) cannot happen. For the remaining 24 permutations the following key may be verified:

light coin

1	2	3	4	5	6	7	8	9	10	11	12
LLL	*LLH*	*LLS*	*LHS*	*HLH*	*HSL*	*HSH*	*HSS*	*SHL*	*SHH*	*SHS*	*SSL*

heavy coin

1	2	3	4	5	6	7	8	9	10	11	12
HHH	*HHL*	*HHS*	*HLS*	*LHL*	*LSH*	*LSL*	*LSS*	*SLH*	*SLL*	*SLS*	*SSH*

Also solved by D. W. Alling, LeRoy Babcock, Murray Barbour, Paul Brock, D. H. Browne, Mannis Charosh, Monte Dernham, Clara Fellers, Harley Flanders, G. E. Forsythe, Frank Herlihy, Vern Hoggatt, J. A. Jenkins, A. E. Karp, Nobert Kaufman, Ralph Keffer, N. D. Lane, H. R. Leifer, C. O. Oakley, Victor Perlo, C. L. Perry, C. F. Pinzka, Joseph Rosenbaum (also like I), Cedric Smith, Waldo Steiner, G. Sved, P. D. Thomas, J. A. Waidelich, Jr., F. M. Wood, and the proposer.

Forsythe and Pinzka gave solutions like II; all other solutions were similar to I. Babcock, Feller, Forsythe, Jenkins, Kaufman, Keffer, Perlo, Smith and Sved gave generalizations. These generalizations will be considered later when the solutions to problem 4203 [1946, 278] are published.

C. D. Olds drew attention to the generalizations of "bad" coin problems published in the *Mathematical Gazette*, 1945, p. 227. Forsythe mentioned the interesting problem: Given 31 coins, of which exactly one is false, being of weight unequal to that of the other 30, and also given a *spring* balance. In five weighings determine (1) which coin is false, (2) the weight of the false coin, (3) the weight of the good coins.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known text books or results found in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4229. *Proposed by Paul Erdös, University of Michigan*

Let $f(z)=z^n+\cdots+a_n$, $g(z)=z^m+\cdots+b_m$ be two polynomials. Denote by A the region where $|f(z)|\leqq 1$ and by B the region where $|g(z)|\leqq 1$. Prove that A cannot properly contain B.

4230. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In every system of numeration in which the base B is divisible by two or more distinct primes but not by 3 or any prime of the form $6k+1$, the numbers which have the property that they are reproduced in the right-hand digits of their squares are the same as those which are reproduced at the right of their fourth powers. (For example: $B=10$, $76^2=5776$, $76^4=33362176$.)

4231. *Proposed by Paul Nemenyi, Washington State College*

Show that any parabola $y=ax^n (a\neq 0,\ n>0)$ has the following property: If through the vertex any ray is drawn, the ratio of the area of the segment to that of the largest inscribed triangle is independent of the direction of the ray. Are there other curves with the same property?

4232. *Proposed by H. D. Ruderman, New York City*

$A:(0, 0)$ and $B:(0, u)$ are joined by the straight line segment AB and a curve π to enclose a region of area s. Let k represent the length of π from A to B. Find the equation of π such that the ratio s/k^2 is a maximum.

4233. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Parallel lines, of arbitrary direction, through the vertices A', B', C', D' of a tetrahedron $A'B'C'D'$ intersect in A_1, B_1, C_1, D_1 the faces BCD, CDA, DAB, ABC of a homothetic tetrahedron $ABCD$. If k is the homothetic ratio, V the volume of $ABCD$, and V_1 that of $A_1B_1C_1D_1$, then

$$V_1 = -k^3(2k+1)V.$$

4206 [1946, 341]. *Corrected. Proposed by Victor Thébault, Tennie, Sarthe, France*

Consider spheres with centers at the vertices of a tetrahedron $ABCD$ and radii equal respectively to k times the sum of the squares of the three opposite

edges. Show that the sum of the squares of the distances from the four spheres to the center of the sphere orthogonal to the four spheres is equal to

$$[2(4k+1)]^2R^2 - 2k(2k+1)\Sigma,$$

where R is the radius of the circumsphere and Σ means the sum of the squares of the six edges. Consider particular cases.

4208 [1946, 342]. *Corrected. Proposed by Victor Thébault, Tennie, Sarthe, France*

Given an orthocentric tetrahedron. If two isogonal conjugate points are also conjugate with respect to the circumsphere, their pedal sphere is orthogonal to the sphere, belonging to the linear net determined by the circumscribed and conjugate spheres, and whose center is the complementary point of the orthocenter with respect to the tetrahedron.

SOLUTIONS

Parabola and Strophoid

4026 [1942, 128]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

(1) Construct a triangle ABC knowing a, A and given that the median and symmedian from A are perpendicular and parallel to two given directions. (2) Indicate the properties of this special triangle. (3) Let B' and C' be the orthogonal projections of B and C on a variable straight line AP which cuts BC in P. The locus of the harmonic conjugate of P with respect to B' and C' is a right strophoid having the vertex A for a double point and tangent to the bisectors of angle A.

Solution by R. Bouvaist, Vincelles, Saône et Loire, France. Construction of the triangle. The segments $A\mu$, $A\sigma$ being drawn parallel to the given directions, the interior bisector of the angle formed is $A\delta$; the two lines $A\beta$ and $A\gamma$ are drawn symmetric to $A\delta$ making with it the given angle $A/2$. Let $A\mu'$ be the harmonic conjugate of $A\mu$ with respect to $A\beta$ and $A\gamma$, draw on $A\mu'$ the segment $AB'=a/2$; then through B' draw the parallel to $A\mu$ meeting $A\beta$ in B; and through B draw the parallel to $A\mu'$ which must meet $A\gamma$ in C, giving the desired triangle ABC.

Characteristic properties when $A\mu$ and $A\sigma$ are perpendicular. Let $A\mu$ and $A\sigma$ be the median and symmedian from A, with $c>b$; we then have

$$\frac{B\mu}{\sin\left(\frac{A}{2}-\frac{\pi}{4}\right)}=\frac{A\mu}{\sin B}, \qquad \frac{C\mu}{\sin\left(\frac{A}{2}+\frac{\pi}{4}\right)}=\frac{A\mu}{\sin C},$$

$$\frac{\sin\left(\frac{A}{2}+\frac{\pi}{4}\right)}{\sin\left(\frac{A}{2}-\frac{\pi}{4}\right)}=\frac{\sin C}{\sin B}=\frac{c}{b}=\frac{\tan\frac{A}{2}+1}{\tan\frac{A}{2}-1}, \qquad \tan\frac{A}{2}=\frac{c+b}{c-b}.$$

Locus of the harmonic conjugate of P with respect to B' and C. Let P' be the harmonic conjugate of P with respect to B and C; the perpendicular from P' to AP meets it in Q, then $QC'PB'$ is harmonic. The envelope of $P'Q$ is tangent to BC (P being the orthogonal projection H of A on BC), to the median for BC and to the line at infinity (P at the midpoint μ of BC), to the bisectors of angle A (P at the foot of one of these bisectors). The envelope is moreover of the second class, since from a point P' of BC there is one and only one tangent; it is therefore a parabola with the median $A\mu$ as the directrix. The locus of Q is the pedal of this parabola with respect to the point A on its directrix, and the locus is a stophoid; with double point at A, the tangents at this point being the bisectors of the angle A, and having for real asymptotic direction the median $A\mu$. If $A\mu$ and $A\sigma$ are perpendicular the point A is the intersection of the directrix and the axis of the parabola, and the strophoid is right angled.

Editorial Note. See the proposer's indications of a solution 1943, p. 333.

Triangles and Conics Circumscribed and Inscribed

4127 [1944, 352]. *Corrected. Proposed by Victor Thébault, Tennie, Sarthe, France*

The straight lines AG, BG, CG, DG drawn through the vertices and centroid G of the tetrahedron $ABCD$ meet again its circumsphere in A', B', C', D', and the planes perpendicular to the respective lines at these latter points determine the tetrahedron $A_1'B_1'C_1'D_1'$. Show that (1) The two tetrahedrons have the same centroid and are hyperbolic. (2) The non-focal axis of the quadric surface with G as a focus inscribed in $A_1'B_1'C_1'D_1'$ is equal to the diameter of the orthoptic sphere of the Monge sphere of the Steiner ellipsoid inscribed in $ABCD$.

Solution by the Proposer. The following generalization will be considered.

I. Let A', B', C' be the points in which the straght lines AP, BP, CP meet again the circumcircle of the triangle $T \equiv ABC$ ($BC=a$, $CA=b$, $AB=c$), where P is arbitrarily chosen. Set $T_1 \equiv A_1B_1C_1$, $T_1' \equiv A_1'B_1'C_1'$, the antipedal triangles of P with respect to T, $T' \equiv A'B'C'$; let P' be the symmetric of P with respect to the center O of the circle (O, R) the circumcircle of T.

The triangles T and T_1' are polar reciprocal with respect to circle (P), center P and radius whose square is the power of P with respect to (O, R). These two triangles, which are orthologic by construction, are also homologic; moreover, P has the same barycentric coordinates with respect to each. These properties are also true for T' and T_1.

The triangles T_1 and T_1' circumscribe the same conic with foci P, P', and the lengths 2α, 2β of its axes are such that $\alpha = R$ and $\beta^2 = R^2 - (OP)^2$. If $P \equiv G$ the centroid of T, it is also the centroid of T_1'. Thus T_1 and T_1' circumscribe a conic with foci G and G', with focal axis $2\alpha = 2R$ and the semi axis β, non-focal, is such that

$$\beta^2 = R^2 - (OG)^2 = (a^2 + b^2 + c^2)/9, \qquad \beta = (a^2 + b^2 + c^2)^{1/2}/3 = \rho\sqrt{2},$$

where ρ is the radius of the Monge circle of the Steiner ellipse inscribed in T.

THEOREM. *In a triangle $T \equiv ABC$, the non-focal axis of the conic inscribed in the antipedal triangle of the centroid G, one focus being G, has the length of the diameter of the orthoptic circle of the Monge circle of the inscribed Steiner ellipse.*

II. The straight lines from the arbitrarily chosen point P to the vertices of the tetrahedron $T \equiv ABCD$ meet again the circumsphere (O, R) of T in A', B', C', D', where $BC = a$ and $DA = a'$, *etc.* Denote by $T_1 \equiv A_1B_1C_1D_1$, $T_1' \equiv A_1'B_1'C_1'D_1'$ the antipedal tetrahedrons of P with respect to T, T'; by P' the symmetric of P with respect to the center O of (O, R). Then T and T_1' are polar reciprocal with respect to sphere (P) with center P and with the radius whose square is the power of P with respect to (O, R). These two spheres are orthologic by construction, and they are also hyperbolic; moreover, P has the same barycentric coordinates for T and T_1'; and these properties are also true for T' and T_1.

Also T_1, T_1' circumscribe the quadric surface of revolution with foci P, P' and the lengths 2α, 2β of the axes of the meridian conic are such that $\alpha = R$, $\beta^2 = R^2 - (OP)^2$.

If $P \equiv G$ the centroid of T, it is also the centroid of T_1'. Thus T_1, T_1' circumscribe the quadric surface of revolution with foci G and G', where the focal axis length $2\alpha = 2R$ and the non-focal semi-axis length β is such that

$$\beta^2 = R^2 - (OG)^2 = \Sigma[a^2 + (a')^2]/16,$$

$$\beta = \{\Sigma[a^2 + (a')^2]\}^{\frac{1}{2}}/4 = \rho\sqrt{3},$$

where ρ is the radius of the Monge sphere of the Steiner ellipsoid inscribed in T. Hence we have

THEOREM. *In a tetrahedron $T \equiv ABCD$, the non-focal axis of the quadric surface of revolution inscribed in the antipedal tetrahedron of the centroid G, one focus being G, has the same length as the diameter of the orthoptic sphere of the Monge sphere of the inscribed Steiner ellipsoid.*

Note 1. In the case of a triangle T, the second focus G' of the conic inscribed in the triangles T_1 and T_1', symmetric of the point G with respect to the center O of the circumscribed circle, is on the straight line joining the Steiner point to the reciprocal of the orthocenter.[1] Moreover, the point G' is such that if circles are described with diameters $G'A$, $G'B$, $G'C$, the sum of the powers of a vertex of T with respect to two of the circles not passing through that vertex is the same for the three vertices.[2]

Note 2. For the hypothesis that $P \equiv G$, the triangle $T' \equiv A'B'C'$ has interesting properties. The construction of a triangle $T \equiv ABC$ knowing the vertices of T' amounts to determining the point G in its plane such that the three vectors carried by GA', GB', GC' of intensities $1/GA'$, $1/GB'$, $1/GC'$ have a zero re-

[1] Mathesis, 1888, p. 178

[2] E. Lemoine, Journal de G. de Longchamps, 1885, p. 219.

sultant. This leads to the determination of the foci of a conic touching the sides of $A'B'C'$ at their midpoints (inscribed Steiner ellipse). It follows also that the foci S and S' of the inscribed Steiner ellipse (E) for T coincide with the centroids of the antipedal triangles T_1 and T_1' of the points S and S' with respect to T and there results the equations

$$(SA)^2 \cdot BC \cdot SS_a = (SB)^2 \cdot CA \cdot SS_b = (SC)^2 \cdot AB \cdot SS_c,$$
$$(S'A)^2 \cdot BC \cdot S'S_a' = (S'B)^2 \cdot CA \cdot S'S_b' = (S'C)^2 \cdot AB \cdot S'S_c',$$

where S_i and S_i' are the orthogonal projections of S and S' on BC, CA, AB. For, SB_1C_1 and BCS' being similar triangles, we have

$$(SA)^2 \cdot BC \cdot SS_a = SA \cdot BC \cdot SS_a \cdot S'S_a' = SB \cdot CA \cdot SS_b \cdot S'S_b' = SC \cdot AB \cdot SS_c \cdot S'S_a'$$

since S is the centroid of T_1 and the products $SS_i \cdot S'S_i'$ are equal to the square of the non-focal semi axis of (E).

Note 3. For the hypothesis that $P \equiv G$, the centroid of the tetrahedron T, the four vectors $GA', \cdots, GD'$ carry vectors of intensities $1/GA', \cdots, 1/GD'$ which have a zero resultant since the tetrahedrons T and T_1' have the same centroid. But the determination of the points G knowing the points $A', \cdots, D'$[3] is subject not to a quadric surface inscribed in T' but to a surface of the third class, the polar reciprocal of the Cayley cubic surface (locus of the points whose orthogonal projections on the faces of the tetrahedron are coplanar).

Self Reciprocal Curves

4167 [1945, 400]. *Proposed by R. A. Staal, Student, University of Toronto*

What curves are self-reciprocal with respect to the conic $x^2+y^2=z^2$ (or the circle $x^2+y^2=1$)?

Note by Howard Eves, Oregon State College. Associated with problem 4167 is problem E 666 [1945, 218]. A solution for E 666 by Francis Hall has already appeared [Jan. 1946, 40]. To this solution of Hall's, I appended the following editorial Note:

"Using homogeneous coordinates it is easy to show that the curve $f(x\ y, z)=0$ will be self-reciprocal to the circle $x^2+y^2=z^2$ if and only if $f(x, y, z)$ divides $f(\partial f/\partial x, \partial f/\partial y, -\partial f/\partial z)$. Imposing this condition on the curve $x^m y^n = kz^{m+n}$ we readily find the values of k as given above."

Remark by the Proposer. The condition for an algebraic curve

$$f \equiv f(x, y, z) = 0$$

to be self-reciprocal with respect to $x^2+y^2=z^2$, seems to be that the polynomial f should divide $f(\partial f/\partial x, \partial f/\partial y, -\partial f/\partial z)$. An instance is

$$f = x^m y^n - kz^{m+n}, \qquad k = \sqrt{m^m n^n/(m+n)^{m+n}}.$$

[3] Thébault, Mathesis, 1939, p. 51 (Sûjet d'étude).

RECENT PUBLICATIONS

EDITED BY H. P. EVANS, University of Wisconsin

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y., and not to any of the other editors or officers of the Association.

The Common Sense of the Exact Sciences. By W. K. Clifford. New York, Alfred A. Knopf, 1946. 66+249 pages. $4.00.

The original work as planned by Clifford was to have been entitled *The First Principles of the Mathematical Sciences Explained to the Non-Mathematical.* This would have been a clumsy title, but more descriptive of its nature than *The Common Sense of the Exact Sciences*, the title finally adopted in accordance with a preference expressed by Clifford shortly before his death in 1879. He left the manuscript unfinished. The labor of revision and completion was begun by R. C. Rowe and finished by Karl Pearson. The first edition appeared in 1885, and this was followed by a second and a third edition, but the book has now been a long time out of print, so that the present edition is very welcome. It is essentially a reproduction of the third edition, the notable additions being a Preface by Bertrand Russell and an Introduction by the Editor, James R. Newman.

Every teacher of mathematics, particularly elementary mathematics, should read this Preface. I would like to quote the whole of it, because it would make a better review of the book than I can write. However, the following quotation must suffice: "A taste for mathematics, like a taste for music, can be generated in some people, but not in others . . . . Pupils who have not an unusually strong natural bent towards mathematics are led to hate the subject by two shortcomings on the part of their teachers. The first is that mathematics is not exhibited as the basis of all our scientific knowledge, both theoretical and practical: the pupil is not convincingly shown that what we can understand of the world, and what we can do with machines, we can understand and do in virtue of mathematics. The second defect is that the difficulties are not approached gradually, as they should be, and are not minimized by being connected with easily apprehended central principles, so that the edifice of mathematics is made to look like a collection of detached hovels rather than a single temple embodying a unitary plan. It is especially in regard to this second defect that Clifford's book is valuable."

The Introduction gives an interesting account of Clifford as man and as scientist. Here are a few brief facts. He was born at Exeter in England in 1845. He studied at King's College, London, and later at Cambridge. "In a rigidly conventional age he was marked by eccentricities of habit, dress, and opinion." In addition to mathematics, he studied French, German, Spanish, Arabic, Greek, Sanskrit, and hieroglyphics. He also learned Morse code and shorthand, and topped his athletic career hanging by his toes from the cross-bar on the weather-

cock of a church steeple. In 1868 he was elected to a fellowship at Trinity, and in 1871 was appointed professor of applied mathematics at University College, London, Clerk Maxwell being one of those recommending him for the position, on the basis that his researches did not tend to "the elaboration of abstruse theorems by ingenious calculation, but to the elucidation of scientific ideas by the concentration upon them of clear and steady thought." He died of tuberculosis in 1879.

There are five chapters with the following titles: Number, Space, Quantity, Position, Motion. These titles give a rough idea of the scope of the book. There is a good deal of material open to criticism on various grounds, but there runs through the whole work a thread of something which, if not pure gold, looks very like it. Would that a similar thread ran through the textbooks of today! Writers of textbooks should ponder the fact that killing the interest of students is an easy task; its stimulation is a much more subtle thing, and they should not be ashamed to learn from a master of exposition in this field. If the textbook writer is an active research mathematician, as Clifford was, he would do well to investigate Clifford's secret of the interest-grasping simile and the complete absence of talking-down to his readers.

To the modern scientist the book is not likely to yield much, directly. But indirectly it may give food for thought. One would have to be a highly specialized historian of science to fill the details in the scientific backdrop before which Clifford performed his intellectual act. But we may imagine it filled with a dignified figure of Mother Nature, holding two scrolls —one with Euclid's axioms and the other with Newton's laws of motion. Only the very observant had noticed that a little worm had already eaten into Euclid's axioms, and only an iconoclast would play with the idea that Newton's laws might suffer a like indignity. But Clifford was observant and he was an iconoclast. He was moreover an admirer of Riemann, and had translated Riemann's now famous paper "On the Hypotheses which lie at the Bases of Geometry." It was left to Einstein to detect the flaw in Newton's concept of time, and so reduce Newton's laws from a position of divine authority to the level of a useful approximation, but the worm that began this process of decay was Clifford's. Science is such a terribly serious thing that it is a relief to know that Clerk Maxwell had a pet demon and Clifford a pet worm; its name was AB, and anyone interested in knowing more about it will find a sketch of it on p. 194 of the volume under review, and a description of its psychological reactions on the subsequent pages. All of which goes to show that the progress of science owes as much to the men who play with new ideas as it does to those who elaborate "abstruse theorems by ingenious calculation."

The publishers are to be congratulated on producing a volume pleasing to the eye and worthy of its author.

J. L. SYNGE

Analytic Geometry and Calculus. Second Edition. By H. B. Phillips. New York, John Wiley and Sons, Inc.; London, Chapman and Hall, Ltd., 1946. 504 pages. $4.50.

In this book analytical geometry and calculus are presented in the form and order in which these subjects are required for courses in science and engineering. The author assumes that it is the calculus that is most used, and introduces analytical geometry only as it is needed to develop the calculus. Chapter I gives a brief but adequate introduction to the ideas of coordinates, functions, graphical representation, limits of sequences and functions including the limit of $\sin\theta/\theta$ as $\theta\to 0$, continuity of functions, the Cauchy criterion of convergence, and uniform continuity. In Chapter II instantaneous speed and the slope of a curve as the limit of the slope of a secant line are studied preliminary to introducing the derivative. The differential is then defined in the following way: The differential of the independent variable is equal to its increment and the differential of the function of the independent variable is equal to the product of its derivative and the increment of the independent variable. There then follows the differentiation of the algebraic functions, and of the sine and the cosine. With this preparation the author takes up the use of the differential in approximation and in determining small errors, derivatives of higher order, implicit functions, maxima and minima, rates of change, velocity and acceleration, mean value theorem, Rolle's theorem, and indeterminate forms. Chapter III deals with integration and summation. The process of finding a function with a given differential is called integration. The definite integral is defined as the limit of a sum, and it is proved that if $F(x)$ is a function such that $dF(x)=f(x)dx$ then the integral of $f(x)$ from $x=a$ to $x=b$ is $F(b)-F(a)$. Integration is then used to obtain areas, volumes, pressures and work. The mean value theorem and the fact that if $F(x)$ is the integral of $f(x)$ from a to x then $dF(x)=f(x)dx$ are proved.

The first three chapters contain, as the author states in the preface, a complete course in the calculus. As to the wisdom of so great a concentration on the calculus at the expense of analytical geometry in the early stages of a course for scientists and engineers, there is room for a difference of opinion. It is a procedure with which the reviewer is in complete agreement. At this stage the students show considerable enthusiasm for the ideas of the calculus. They are finding a use for them in their courses in physics, chemistry, and engineering, and it is a wise policy to exploit this interest.

In Chapters IV, V, and VI the author takes up the straight line, Newton's method, conic sections, transformation of coordinates, graphs of algebraic functions, determinants and their application to linear equations, and trigonometric functions. Chapter VII returns to the calculus with the differentiation and integration of the exponential and logarithmic functions, and from this point on the topics follow fairly closely the established tradition in elementary calculus, with one notable exception. The notation of vector analysis is developed in two and three dimensions, and this is used in finding centroids, in studying velocity and acceleration, tangents and normals to curves in two and three dimensions,

angular velocity, tangent planes and normals to a surface, line integrals and gradients. It is the reviewer's opinion that this innovation is long overdue.

Enough has now been said to indicate that this text is something new in the field of elementary calculus. There is a fresh approach to every topic. Moreover, as one reads the book it becomes clear that the changes from tradition are not made merely for the sake of being different, but because long experience has convinced the author that this is the way to give to scientists and engineers the mathematical training that will be most useful to them. In regard to this the reviewer would raise only one question: Should the student go as far with the calculus as this book carries him without becoming acquainted with elementary differential equations?

R. L. JEFFERY

A Manual of Operation for the Automatic Sequence Controlled Calculator. (Annals of the Computation Laboratory of Harvard University, Vol. I.) By the Staff of the Computation Laboratory, Harvard University. Cambridge, Harvard University Press, 1946. 561 pages. $10.00.

The Staff of the Harvard Computing Laboratory has undertaken in this Manual, the difficult task of explaining to the uninitiated the construction and use of the elaborate Harvard Sequence Controlled Calculator. As its title implies, the Manual is intended primarily as a set of instructions for operators of the Calculator; but several chapters will be of value both to mathematicians who have problems suitable for machine computation and to designers of automatic computers.

The parts of the Manual which will be of interest to the more extensive class of readers are the first 52 pages, including a brief historical sketch of digital calculators in Chapter I and a description of the Calculator in Chapter II, and the 66-page bibliography of computing machines and techniques which contains over a thousand references. The bibliography is divided into 23 sections dealing with such subjects as Historical Background of Automatic Calculating Machinery, Determinants, Zeros of a Polynomial, Harmonic Analysis, Interpolation, Asymptotic Expansions, Numerical Differentiation and Integration, Ordinary and Partial Differential Equations and Integral Equations.

The Harvard Sequence Controlled Calculator, built by the International Business Machine Corp. under the guidance of Prof. Howard H. Aiken, is a major engineering project. It is an assemblage of calculating and control elements, mounted on racks 8 feet high and totalling 63 feet in length, and weighing about 5 tons. A 4-horsepower motor furnishes its mechanical power through a network of shafts and gears. Sixteen photographs in the Manual show views of the machine as a whole and of various components. One photograph of a computing element of the early (1834) Babbage Difference Engine—a forerunner of modern automatic calculators—contrasts the mechanisms available to designers in the past with those used in the Sequence Calculator.

The Calculator has been operated for nearly two years by the Staff of the Harvard Computing Laboratory which, under the editorship of Lt. Grace M.

Hopper, USNR, has compiled this Manual. As a kind of side line to its regular confidential war work in this period, it has calculated ab initio at least six volumes of mathematical tables one of which—a 230 page table of Modified Hankel Functions—has been published. In these computations the Calculator normally uses 23 digital places but it can, under certain conditions, deal with numbers of 46 or of 12 digits. Prof. Aiken estimates that the machine will do the work of about 100 well equipped manual computers.

Chapter II of the Manual contains a general description of the Calculator and of its major component parts. Functionally, the Calculator contains the components common to almost all automatic computing devices, namely: means for accepting formulas and numerical data; for transferring and storing numbers; for performing standard arithmetic operations, such as addition, multiplication, and so forth, on these numbers; and finally, for presenting the results of the computations.

The storage, addition and transfer of numbers in the Calculator are carried out by methods and devices which are familiar to those who have used the standard International Business Machine equipment, although such familiarity on the part of the reader is not assumed in the Manual. The novel features of the Calculator reside principally in the control of the machine.

The Sequence Controlled Calculator has, like Babbage's Engines and like most automatic computing devices since that time, been conceived as an automatic means for carrying out a sequence of mathematical operations on any set of numbers which may be presented to it. The sequence of operations is derived by a human process of "coding" from mathematical formulas (a power series of a Bessel Function, for example). The control sequence is entered into the machine separately from the numerical data, so that it may be used over and over with many different sets of numerical data.

The numbers which the Calculator is instructed to operate on are coded and the codes are punched either in standard IBM cards or on a continuous tape. Certain constants may be set manually on switches, there being one 10—position switch for each digit to be stored. In all, the Calculator contains two punched-card readers, one "value" tape (exclusive of "interpolation" tapes) and sufficient switches to store 60 numbers of 23 digits each.

Instructions to the machine consist of repetitions of the command "take the number from unit A, deliver it to unit B and start operation C," the codes for A, B and C being punched in the sequence control tape. The unit A may be any one of the sources of numerical data or any one of the 72 adding-storage units in the Calculator. The unit B must, of course, be a unit which, like the adding-storage units, is capable of receiving a number.

Three interpolating units are built into the machine. Each such unit is supplied with a tape and each is capable of interpolating up to the 11th order between the numerical values punched in its tape. Besides these units, electromechanical tables of $\sin A$, $\log A$ and 10^x, are permanently wired into the Calculator. Internal sequence controls permit the main control tape to call for

interpolation from any one of these tables. Other operations available are addition, multiplication, division, reading of absolute values, and so forth.

The results of computations are presented automatically, either on IBM punched cards or on one or both of the two electric typewriters with which the Calculator is equipped.

Some form of mathematical check on the computation is essential to reliable operation of the Calculator. A special counter is provided which will stop the machine should the check computation indicate an error in the results.

To provide for flexibility in the location of decimal points and in the number of digits used in multiplication and division, a plugboard has been included in the Calculator. Chapter V of the Manual is devoted to special instructions for using the plugboard.

A number of examples of problems are worked out in Chapter VI. These include the computation and checking of a quartic polynomial, of $(x^2-1)^{-3/2}$ by an iterative process, of an integral, of the solution of a system of linear algebraic equations and other problems.

Mechanisms which will perform the fundamental arithmetic operations involved in computation have been improved and simplified by the intensive study of hundreds of philosophers, scientists and inventors and by the construction and practical use of thousands of models since their crude beginnings in the 17th century, but mechanisms which will carry out the analogous duties for the "programming" of sequences of such operations have not as yet had the benefit of a comparable background. The early stages in the development of any class of mechanisms are characterized by relatively clumsy and groping attempts to satisfy conflicting requirements, and it is no reflection on the ingenuity and capability of the Harvard Computing Laboratory Staff to point out that this generalization applies to the Sequence Control of the Calculator. It is not a simple or easy task to learn to set up the controls for the Calculator, but it is to be hoped that continued development of the machine will lead to simpler and easier procedures in this important part of the field of automatic computation. In fact, the existence of the Calculator should, in itself, furnish both an incentive to devise such improvements and an opportunity to try them out under conditions of actual use. The present methods of coding or translating from mathematical symbols to machine language are given in some detail in Chapter IV, and are illustrated in the examples worked out in Chapter VI. The reader will be impressed by the dissimilarity of the two languages, and will probably conclude that the translation from one to the other had best be left to the experts in the process.

G. R. STIBITZ

Analytische Geometrie der Ebene und des Raumes. (Lehrbücher und Monographien aus dem Gebiete der Exakten Wissenschaften, No. 4; Mathematische Reihe, Band II). By Rudolf Fueter. Basel, Birkhäuser, 1945. 180 pages. Pamphlet bound, 18.50 s. fr.; cloth bound, 22.50 s. fr.

The chapter headings are as follows: Chapter I (pp. 11–51), Point and Line

in the Plane. Chapter II (pp. 52–94), Point, Plane and Line in Space. Chapter III (pp. 95–142), Curves of the Second Degree in the Plane (Conic Sections). Chapter IV (pp. 143–172), Surfaces of the Second Degree.

The book is elementary in character and treats the material which is usually presented in a first course in analytic geometry. In the first twenty-one pages of the text most of the simple tools for the study of plane analytic geometry are presented, such as; rectangular (cartesian) coördinates and polar coördinates, including the relations between them, equations of translations and rotations, determinants and vectors. Even though vectors are introduced no systematic use of them is made later on. It is somewhat disappointing that a third order determinant is expanded by the device of rewriting the first two columns.

In Chapter IV the quadric surfaces are considered with their equations in "standard forms." The problem of reducing the general second degree equation to one of the standard forms is mentioned but not treated.

It is the opinion of the reviewer that the book is well written. The ideas are carefully explained and the discussions are thorough. This is a nice book for the undergraduate to use for practice in reading German. Finally, the typography is excellent.

J. H. TAYLOR

NEW BOOKS RECEIVED

Concise Analytic Geometry. By C. H. Sisam. New York, Henry Holt and Co., 1946. 9+155 pages. $2.00.

Applied Elasticity. By John Prescott. New York, Dover Publications, 1946. 6+666 pages. $3.95.

Higher Mathematics for Students of Chemistry and Physics. Fourth Edition. By J. W. Mellor, New York, Dover Publications, 1946. 21+641 pages. $4.50.

Elementary Matrices and Some Applications to Dynamics and Differential Equations. By R. A. Frazer, W. J. Duncan, and A. R. Collar. New York, Macmillan Co.; Cambridge University Press, 1946. 16+416 pages. $4.00.

Grundebegriffe der Wahrscheinlichkeitsrechnung. By A. Kolmogoroff. New York, Chelsea Publishing Co., 1946. 5+62 pages. $2.25.

Grundlagen der Analysis. (Foreword translated into English and vocabulary added). By Edmund Landau. New York, Chelsea Publishing Co., 1946. 20+139 pages. $2.50.

Lectures on the Calculus of Variations. By G. A. Bliss, Chicago, University of Chicago Press, 1946. 9+296 pages. $5.00.

Mathematical Methods of Statistics. By Harald Cramér. Uppsala, Almqvist and Wiksells, 1945; Princeton University Press, 1946. (Princeton Mathematical Series, No. 9). 16+575 pages. $6.00.

Mathematics of Finance. By J. A. Northcott. New York, Rinehart and Co., 1946. 10+252 pages. $3.00.

Practical Electrical Mathematics. By W. E. Rasch. Boston, D. C. Heath and Co., 1946. 8+360 pages. $2.00.

Reelle Funktionen. Vol. 1. *Zahlen, Punktmengen, Funktionen.* By Constantin Carathéodory. New York, Chelsea Publishing Co., 1946. 6+184 pages. $3.25.

Scientific and Technical Aspects of the Control of Atomic Energy. (United Nations Department of Public Information.) New York, Columbia University Press, 1946. 5+42 pages. $0.25.

Tables of Fractional Powers. Prepared by the Mathematical Tables Project, National Bureau of Standards. New York, Columbia University Press, 1946. 489+30 pages. $7.50.

Trigonometry Refresher for Technical Men. By A. A. Klaf. New York and London, McGraw-Hill Book Co., Inc., 1946. 10+629 pages. $5.00.

Vorlesungen über Zahlentheorie aus der Elementaren Zahlentheorie. By Edmund Landau. New York, Chelsea Publishing Co., 1946. 8+184 pages.

CLUBS AND ALLIED ACTIVITIES

EDITED BY L. F. OLLMANN, Hofstra College

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to L. F. Ollmann, Hofstra College, Hempstead, New York.

Honor societies have been in existence for many years, and men and women of high scholastic standing have coveted membership in such societies. Departmental honor fraternities are of rather recent origin, but are now found on most campuses. Two national mathematical fraternities are serving mathematicians in the United States. Their growth and progress have demonstrated the need and value of affiliation of members of a mathematical club at one institution with those of similar organizations at other institutions. Some of the advantages gained by such an affiliation include: a recognized standard in honoring students of unusual ability in mathematics, the encouragement of student participation in undergraduate as well as graduate research, and the bringing together of students and faculty in fraternal relations with those of other campuses.

In an endeavor to acquaint mathematicians with the history and purpose of these two fraternities and to solicit further interest in them, this Department presents in this issue of the MONTHLY some descriptive material about *Pi Mu Epsilon* and *Kappa Mu Epsilon.* If any active mathematical club is interested in affiliating with one of the two societies, it is suggested that the faculty member in charge of the club correspond with the national secretary of the fraternity in which the group is interested.

PI MU EPSILON

Pi Mu Epsilon, the older of the two societies, was founded by Professor E. D. Roe, Jr., a member of the mathematics faculty at Syracuse University. It was incorporated in 1914 under the laws of the State of New York. Professor Roe remained Director General of the fraternity during the remainder of his life.

The purpose of *Pi Mu Epsilon*, as stated by Director General Tomlinson Fort, is "to serve mathematicians on the campuses of our universities by bringing together in a national organization those faculty, graduate students, and undergraduate students who are interested in Mathematics." All chapters hold mathematical meetings similar to those of the usual mathematics clubs.

Many new activities have been sponsored by chapters of *Pi Mu Epsilon:* the holding of mathematical contests, the awarding of mathematical prizes, the sponsoring of mathematical lectures by distinguished speakers, and the installation of mathematics clubs both in their own institutions and in high schools. For a number of years the chapter at New York University sponsored a state-wide mathematics contest among secondary schools with the greatest success. Nothing is secret concerning *Pi Mu Epsilon*; guests are frequently present at meetings and at initiations.

The National Officers of *Pi Mu Epsilon* are:

Director General:	Tomlinson Fort, University of Georgia
Vice-Director General:	E. H. C. Hildebrandt, Northwestern University
Secretary-Treasurer General:	John S. Gold, Bucknell University
Councillors General:	George Williams, Oregon State College
	C. A. Hutchinson, University of Colorado
	C. H. Richardson, Bucknell University
	E. R. Smith, Iowa State College

The name, location, and date of founding of chapters follow:

New York Alpha	Syracuse University	1914
Ohio Alpha	Ohio State University	1919
Pennsylvania Alpha	University of Pennsylvania	1921
Missouri Alpha	University of Missouri	1922
Alabama Alpha	University of Alabama	1922
Iowa Alpha	Iowa State College	1923
Illinois Alpha	University of Illinois	1924
Pennsylvania Beta	Bucknell University	1925
Montana Alpha	Montana State University	1925
New York Beta	Hunter College	1925
Missouri Beta	Washington University	1925
California Alpha	University of California	1925
Ohio Beta	Ohio Wesleyan University	1927
Kentucky Alpha	University of Kentucky	1927
Nebraska Alpha	University of Nebraska	1928
Kansas Alpha	University of Kansas	1928
Pennsylvania Gamma	Lehigh University	1929
Oklahoma Alpha	University of Oklahoma	1929
California Beta	University of California	1930
Pennsylvania Delta	Pennsylvania State College	1930
Arkansas Alpha	University of Arkansas	1931
Oregon Alpha	University of Oregon	1931
Washington Alpha	State College of Washington	1931
North Carolina Alpha	Duke University	1932
Washington Beta	University of Washington	1932
New York Gamma	Brooklyn College	1933

Wisconsin Alpha	Marquette University	1933
New York Delta	New York University	1933
Georgia Alpha	University of Georgia	1934
New York Epsilon	St. Lawrence University	1935
Kansas Beta	Kansas State College	1935
Ohio Gamma	University of Toledo	1936
Colorado Alpha	University of Colorado	1936
New York Zeta	Columbia University	1937
Oklahoma Beta	Oklahoma Agricultural and Mechanical College	1938
Oregon Beta	Oregon State College	1938
Wisconsin Beta	University of Wisconsin	1939
Louisiana Alpha	Louisiana State University	1939
Michigan Alpha	Michigan State College	1940
Arizona Alpha	University of Arizona	1941
Delaware Alpha	University of Delaware	1941
Illinois Beta	Northwestern University	1944
Missouri Gamma	St. Louis University	1945

KAPPA MU EPSILON

Kappa Mu Epsilon was founded by Dr. Kathryn Wyant, a member of the mathematics faculty at Northeastern Oklahoma State Teachers College, in 1931. However, the need for a society in mathematics which would devote all of its interests to undergraduate students seems to have developed simultaneously in many parts of the United States during the years 1929 to 1931. Two colleges in Mississippi, Mississippi State College for Women and Mississippi State College, as well as the University of New Mexico, had shown considerable interest in such a society. These three chapters immediately joined Dr. Wyant in the formation of a national mathematics fraternity of collegiate rank.

The purpose of the fraternity as stated by Professor E. R. Sleight, the National President of Kappa Mu Epsilon, is "to further interest in mathematics and to develop appreciation for the beauty of it, to provide a society for the recognition of unusual ability in mathematics, and to bring them together in fraternal relationship. It encourages undergraduate research and collects information concerning the applications of mathematics."

In the fall of 1941, under the leadership of Professor C. V. Newsom, then National President, *Kappa Mu Epsilon* embarked upon a new enterprise to further undergraduate work in Mathematics. An official journal, *The Pentagon*, was established in order that the fraternity might carry out some of its plans and purposes. It was the belief of the editors that there is a place for a magazine which caters to the needs of college students interested in mathematics, and which features research papers on the level of undergraduate students. These plans and ideals have been realized and in each issue of *The Pentagon* there appear papers by undergraduates.

An initial step toward encouraging college students to attempt something beyond the ordinary routine of class room procedure was taken by the *Michigan*

Alpha Chapter of *Kappa Mu Epsilon* at Albion, Michigan. During the Spring of 1940, invitations were sent to the colleges and universities of the state to meet with that chapter for the purpose of organizing a Michigan Undergraduate Association in Mathematics. Papers were solicited for the meeting and a program by eight students from various Michigan institutions was presented. For two years following this initial effort, very successful meetings were held, one at the Michigan State Normal College at Ypsilanti and one at the Central Michigan College of Education at Mount Pleasant. The war made it impossible to continue but plans have been underway for a meeting at Michigan State College.

The National Officers of *Kappa Mu Epsilon* are:

President:	E. R. Sleight, Albion College
Vice-President:	F. W. Sparks, Texas Technological College
Secretary:	E. Marie Hove, Hofstra College
Treasurer:	L. F. Ollmann, Hofstra College
Historian:	Sister Helen Sullivan, O.S.B., Mount St. Scholastica College

The name, location, and date of founding of chapters follow:

Oklahoma Alpha	Northeastern State College	1931
Iowa Alpha	Iowa State Teachers College	1931
Kansas Alpha	Pittsburg State Teachers College	1932
Missouri Alpha	Southwestern Teachers College	1932
Mississippi Alpha	State College for Women	1932
Mississippi Beta	Mississippi State College	1932
Nebraska Alpha	Wayne State Teachers College	1933
Illinois Alpha	State Normal University	1933
Kansas Beta	Emporia State Teachers College	1934
Alabama Alpha	Athens College	1935
New Mexico Alpha	University of New Mexico	1935
Illinois Beta	Charleston State Teachers College	1935
Alabama Beta	State Teachers College	1935
Louisiana Alpha	Louisiana State University	1936
Alabama Gamma	Alabama College	1937
Ohio Alpha	Bowling Green State University	1937
Michigan Alpha	Albion College	1937
Missouri Beta	Warrensburg State Teachers College	1938
South Carolina Alpha	Coker College	1940
Texas Alpha	Texas Technological College	1940
Texas Beta	Southern Methodist University	1940
Kansas Gamma	Mount St. Scholastica College	1940
Iowa Beta	Drake University	1940
New Jersey Alpha	Upsala College	1940
Ohio Beta	College of Wooster	1941
Tennessee Alpha	Tennessee Polytechnic Institute	1941
New York Alpha	Hofstra College	1942
Michigan Beta	Central Michigan College of Education	1942
Illinois Gamma	Chicago Teachers College	1942
New Jersey Beta	Montclair State Teachers College	1944
Illinois Delta	College of St. Francis	1945
Michigan Gamma	Wayne University	1946

NEWS AND NOTICES

EDITED BY B. W. JONES, Cornell University

Readers are invited to contribute to the general interest of this department by sending news items to B. W. Jones, White Hall, Cornell University, Ithaca, New York.

POSTDOCTORAL FELLOWSHIP OF SIGMA DELTA EPSILON

Sigma Delta Epsilon, Graduate Women's Scientific Fraternity, regularly offers a postdoctoral fellowship amounting to $1500. Applications for the year, 1947–48, should be submitted before February 1, 1947, to the Fellowship Board authorized to make the award.

Women with the equivalent of a Ph.D. degree, carrying on research, in the mathematical, physical, or biological sciences, who need financial assistance and give evidence of high ability and promise are eligible. During the term of her appointment the appointee must devote the major part of her time to the approved research project, and not engage in other work for remuneration (unless such work shall have received the written approval of the Board before the award of the fellowship).

Application blanks may be secured from Dr. Louise S. McDowell, 28 Dover Road, Wellesley 81, Mass. Announcement of the award will be made early in March.

PERSONAL ITEMS

Lehigh University makes the following announcements: Assistant Professors F. S. Beale, E. H. Cutler and A. E. Pitcher have returned from leaves of absence; J. O. Chellevold and R. S. Wentworth have been appointed to assistant professorships.

The United States Naval Academy makes the following announcements: Assistant Professors H. C. Ayres, N. H. Ball, A. E. Currier and J. R. Hammond have been promoted to associate professorships; the following have been appointed to associate professorships: Assistant Professor R. P. Bailey of Lafayette College and Assistant Professor L. H. Chambers of Marshall College; the following to assistant professorships: J. M. Holme, J. P. Hoyt, J. F. Milos, Dr. K. L. Palmquist, Dr. J. F. Paydon, Dr. V. N. Robinson, Dr. S. S. Saslaw.

Professor W. E. Anderson of Miami University has retired as head of the department but is continuing to teach on a part-time basis.

Assistant Professor L. Virginia Carlton of Wesleyan College, Macon, Georgia, has been appointed to an assistant professorship at Centenary College.

G. S. Cook has been appointed to an assistant professorship at Colorado School of Mines.

Dr. Paul Erdös has been appointed to a research professorship at Syracuse University.

W. W. Gandy has been appointed to an associate professorship at Northwestern State College, Louisiana.

Mary A. Goins has been appointed to an assistant professorship at Marshall College.

Associate Professors L. H. McFarlan and A. H. Taub of the University of Washington have been promoted to professorships.

Assistant Professor Thirza A. Mossman of Kansas State College has been promoted to an associate professorship.

F. C. Mosteller has been appointed lecturer in the department of social relations of Harvard University.

Professor C. A. Reagan of Friends University, Wichita, Kansas, has been appointed acting president.

J. K. Reckzeh of the University of Kentucky has been appointed to an assistant professorship at State Teachers College, Jersey City, New Jersey.

Professor J. B. Rosenbach of Carnegie Institute of Technology has been appointed assistant head of the department of mathematics.

Professor L. W. Sheridan of the College of Mount St. Vincent has been appointed to an associate professorship at the College of St. Thomas, St. Paul, Minnesota.

Professor J. A. G. Shirk of Kansas State Teachers College, Pittsburg, Kansas, has retired as head of the department but will continue his teaching duties. Professor R. G. Smith has been appointed head of the department.

Dr. K. H. Stahl has been appointed assistant professor of engineering mathematics at the University of Colorado.

Professor W. H. Watson of the University of Saskatchewan has resigned to accept an appointment with the Canadian Government.

Professor J. H. Zant of Oklahoma Agricultural and Mechanical College has been appointed director of instruction of the Okmulgee Branch, newly established to give instruction to veterans.

The following appointments to instructorships are announced:

The University of Buffalo: Mrs. Joan S. Anderson, V. N. Behrns, Mrs. Jeanne J. Dinwoodie, Mrs. Lorraine W. Farber, Lillian Gough, June M. McArtney, Mabel D. Montgomery, N. H. Sampson, F. C. Warner, Mrs. Ina W. Welmers

Lehigh University: L. Benson, W. Hibbard, H. A. Seebald, R. H. Spohn, K. C. Walters, R. W. Young

Mississippi Southern College: John Jones, Jr., J. T. Lewandowski

Union College: H. K. Holt, E. F. Gillette, E. F. Ormsby.

The United States Naval Academy: M. V. Gibbons, E. C. Gras, T. A. Lamke, Joseph Milkman, J. W. Popow, R. W. Rector, J. A. Tierney, E. C. Watters, J. H. White

The University of Colorado (engineering mathematics): George Barnes, D. L. Barrick, F. J. Casey, R. H. Glass, E. W. Grigs, P. F. Hultquist, V. J. Moore, L. W. Rutland, Jr., M. E. Sperline

The University of Maine: Dr. W. B. Caton

Professor Dunham Jackson of the University of Minnesota died November 6, 1946. He was a charter member of the Mathematical Association.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

ANNUAL MEETING OF THE KANSAS SECTION

The thirty-first annual meeting of the Kansas Section of the Mathematical Association of America was held at the Kansas State Teachers College in Emporia on Saturday, April 13, 1946. The morning session was a joint meeting with the Kansas Association of Teachers of Mathematics. Mr. Edison Greer, Chairman of the Section, presided at both the morning and afternoon sessions.

The attendance was one hundred and fifteen including the following thirty-four members of the Association: Sister M. Nicholas Arnoldy, Wealthy Babcock, W. D. Bemmels, Florence Black, L. E. Curfman, Lucy T. Dougherty, Paul Eberhart, Sister Ann Elizabeth, W. H. Garrett, Laura Z. Greene, Edison Greer, J. R. Hanna, J. O. Hassler, A. J. Hoare, Emma Hyde, W. C. Janes, H. E. Jordan, C. F. Lewis, Anna Marm, Sister Jeannette Obrist, O. J. Peterson, P. S. Pretz, G. B. Price, C. B. Read, J. A. G. Shirk, G. W. Smith, R. G. Smith, E. B. Stouffer, W. T. Stratton, C. B. Tucker, Gilbert Ulmer, E. B. Wedel, J. J. Wheeler, Ferna E. Wrestler.

At the business meeting the following officers were elected for next year: Chairman, C. A. Reagan, Friends University; Vice-Chairman, Sister M. Helen Sullivan, Mount St. Scholastica College; Secretary-Treasurer, Anna Marm, Bethany College.

The following papers were presented:

1. *Varying definitions of mathematical terms*, by Professor C. B. Read and Professor J. R. Hanna, University of Wichita.

The authors presented a survey of recent representative texts, illustrating situations in which authorities use ambiguous or incomplete definitions of mathematical terms, fail to state limitations or assumptions made, or fail to point out the existence of alternative definitions. Illustrative concepts included mantissa, asymptote, principal values of inverse trigonometric functions, definitions of terms common to mathematics and related fields, order of operations, exponents, and significant figures. Those interested in further details may obtain the complete study by requesting *University Studies, Number 17*, from the University of Wichita.

2. *Mathematics for women*, by Sister M. Helen Sullivan, Mount St. Scholastica College.

The aim of the paper is to convey a basic, educational attitude which the writer feels is missing from the present day philosophy of teaching mathematics. The speaker's thesis is this—that since the majority of women are destined to be homemakers, our approach in the teaching of mathematics in women's institutions must be entirely different from that heretofore employed. We have erred in using text books and other devices that cater to the tastes and

interests of men. Mathematics has much to offer in the development of a well-rounded feminine personality. It is the task of teachers of mathematics in women's schools to employ all the forces of mathematics in the training of women.

3. *Common factors in college and high school methods of teaching mathematics*, by Professor J. O. Hassler, University of Oklahoma.

Professor Hassler discussed five teaching problems shared in common by college and high school teachers of mathematics that need to be solved for successful teaching: (1) teaching how to interpret in symbols a quantitative situation expressed in words; (2) teaching students to understand the meaning and use of symbols; (3) securing proficiency in the mechanical manipulation of mathematical symbols; (4) developing and utilizating the power of imagination; and (5) teaching for transfer.

4. *College entrance tests in mathematics*, by Professor W. T. Stratton, Kansas State College, and Dean E. B. Stouffer, University of Kansas.

Professor Stratton gave a brief account of the development of the testing program at Kansas State, and presented considerable data to show that the program is well worth while. The plan of procedure followed at present is to have the freshmen tests given during freshmen week, the papers graded, and the results in the hands of the assigners before the students are assigned. Students failing to come up to a certain standard are placed in a non-credit course their first semester without reference to the amount of high school credit they presented. The objectives to be gained by this program are: (1) to reclaim those who have not had a chance in high school to get the subject, and (2) to separate this failing group from the regular classes so that a higher standard of work can be attained, and so as to place the lower group where they may avoid the stigma of failing in case they have the ability to do college work. Dean Stouffer said that beginning with the present academic year the Department of Mathematics of the University of Kansas has used entrance tests to supplement the information on high school records in order to place freshmen in the courses they are prepared to carry successfully. He stated that the number of students who were required to repeat courses does not greatly exceed the number who were permitted to enter courses in advance of those they would normally enter. The plan will be continued and further reports will be given.

5. *Some mathematical considerations of supersonic flight*, by Professor C. B. Tucker, Kansas State Teachers College.

The speaker discussed the methods available to determine the behavior of an airfoil moving at speed greater than sound. The definitions of "shock wave" and "Mach angle" were introduced. Using the necessary assumptions, the coefficients of lift, drag and moment, and the center of pressure were developed for a double wedge airfoil by the Ackeret theory. These coefficients were compared to those of other airfoils and the results of wind-tunnel tests.

6. *Application of mathematical statistics to agricultural experimentations*, by H. C. Fryer, Kansas State College, introduced by the Secretary.

A discussion was given of certain types of data which require a normalizing transformation before the analysis of variance can be used. An experiment involving tests of cattle fly sprays was used for illustration.

7. *Report on board of governors meeting*, by Professor G. W. Smith, University of Kansas.

Professor Smith gave a report of the meetings of the board of Governors of the Association which were held in Chicago in November 1944 and 1945. He reviewed briefly the formation of the Board and explained the nature and the extent of its work.

ANNA MARM, *Secretary*

ANNUAL MEETING OF THE MICHIGAN SECTION

The annual meeting of the Michigan Section of the Mathematical Association of America was held at the University of Michigan at Ann Arbor on Saturday, April 13, 1946. This meeting also constituted the meeting of the Mathematics Section of the Michigan Academy of Science, Arts and Letters. Morning and afternoon sessions and a luncheon-business meeting were held, at all of which the Chairman, Professor J. W. Bradshaw, presided.

About seventy persons attended the meeting including the following thirty-six members of the Association: H. M. Ackley, J. F. Arena, J. W. Baldwin, W. D. Baten, W. M. Borgman, J. W. Bradshaw, R. V. Churchill, C. J. Coe, A. H. Copeland, C. C. Craig, P. S. Dwyer, C. M. Erikson, C. H. Fischer, K. W. Folley, J. W. Foust, J. S. Frame, V. G. Grove, G. E. Hay, Fritz Herzog, T. H. Hildebrandt, L. A. Hopkins, E. E. Ingalls, L. S. Johnston, P. S. Jones, Wilfred Kaplan, Theodore Lindquist, E. D. Rainville, C. C. Richtmeyer, E. H. Rothe, L. J. Rouse, T. R. Running, E. R. Sleight, T. H. Southard, G. G. Speeker, B. M. Stewart, R. L. Wilder.

At the business meeting the nominating committee, consisting of Professors H. M. Ackley, W. D. Baten and K. W. Folley, nominated Professor E. E. Ingalls of Albion College as Chairman and Professor L. J. Rouse of the University of Michigan as Secretary-Treasurer, and these nominees were unanimously elected.

At the morning and afternoon sessions the following program of eight papers was presented.

1. *Four great mathematicians*, by Professor E. R. Sleight, Albion, College.

Professor Sleight presented a review of the life, work and influence of four early British mathematicians. This paper has been published in the *National Mathematics Magazine*.

2. *Representation of orbits in the restricted problem of three bodies*, by Professor G. P. Loweke, Wayne University, introduced by Professor A. L. Nelson.

By a suitable transformation the speaker set up a correspondence between the orbits of the infinitesimal body and a one parameter family of surfaces in space, and studied the orbits in the light of this representation. This paper was published as the speaker's doctoral dissertation, University of Berlin, 1936.

3. *A problem concerning orthogonal trajectories*, by Professors Fritz Herzog and C. P. Wells of Michigan State College.

The authors considered two families of curves in the plane, namely $\phi(x, y) = C$ and $\psi(x, y) = C$, the two families being orthogonal, and the functions ϕ and ψ both harmonic. The two families are then said to possess property E with respect to the second family if the arc lengths along all curves of the second family are divided proportionally by curves of the first family. The authors showed that if the property holds with respect to the second family, then the property is likewise enjoyed with respect to the first family. They then showed that all families possessing this property fall into four cases, two trivial and two more complicated.

4. *On theorem* 23, *book* 1, *Menelaus Sphaerics*, by Donat Kazarinoff, University of Michigan, introduced by the Secretary.

Mr. Kazarinoff criticized the existing proof of the theorem and offered an amended proof.

5. *Transfinite ordinals*, by Professor K. W. Folley, Wayne University.

Professor Folley's paper was an extension of that of Ben Dushnik published in the *Bulletin of the American Mathematical Society*, vol. 37, 1931. He showed that the Dushnik theorem could be extended to some, but not all, numbers of the second kind.

6. *Conditional invariants and quadric surfaces*, by Professor G. Y. Rainich, University of Michigan.

A conditional invariant of a form (tensor) is an expression in terms of the coefficients of the form (components of the tensor) which remains invariant under a transformation of coordinates if, and only if, certain invariants of the form (tensor) vanish. Professor Rainich showed how such conditional invariants characterize geometric properties of degenerate forms. As a simple example he found the expression for the distance between two parallel lines in terms of the coefficients of the single equation representing the two lines.

7. *Space drawings with a trimetric ruler*, by Professor J. S. Frame, Michigan State College.

The trimetric ruler is an invention of Professor Frame greatly facilitating the drawing of solid figures in parallel perspective. It consists of a scalene triangle of cardboard or plastic with a scale on each edge. Professor Frame developed the theory of the ruler and illustrated its use.

8. *On a certain system of congruences*, by Professor Ben Dushnik, University of Michigan, introduced by the Secretary.

The speaker showed that the problem of solving the equation

$$\frac{x}{y}+\frac{y}{z}+\frac{z}{x}=n$$

in positive integers is equivalent to finding the points of the surface $x^3+y^3+z^3-kxyz=0$, $(k=1, 2, 3, \cdots)$, having positive integral coordinates. He presented several simple examples.

L. J. ROUSE, *Secretary*

MAY MEETING OF THE WISCONSIN SECTION

The fourteenth annual meeting of the Wisconsin Section of the Mathematical Association of America was held at Mount Mary College, Milwaukee, on Saturday, May 4, 1946. Sister Mary Felice presided at the morning session, and Professor Morris Marden presided at the afternoon session.

There were sixty-nine in attendance including the following twenty-six members of the Association: L. K. Adkins, K. J. Arnold, R. H. Bardell, Leon Battig, Ethelwynn R. Beckwith, May M. Beenken, W. W. Bigelow, R. H. Bruck, H. H. Conwell, Sister Mary Felice, J. D. Fitzpatrick, Fannie Hopkins, R. C. Huffer, J. F. Kenney, S. C. Kleene, C. C. MacDuffee, Morris Marden, A. C. Moeller, G. A. Parkinson, H. P. Pettit, Abraham Spitzbart, P. L. Trump, J. I. Vass, R. D. Wagner and Louise A. Wolf.

At the business meeting the following officers were elected for the coming year: Chairman, H. P. Evans, University of Wisconsin; Program Committee, L. K. Adkins, LaCrosse State Teachers College, Louise A. Wolf, University of Wisconsin in Milwaukee (Chairman). The election of R. C. Huffer, Beloit College, as governor of Region Number Nine was announced. A committee of the section was authorized to push the cause of mathematical education and to maintain contact with the coördinating committee of the national association. It was voted to hold the next meeting in May, 1947, at the University of Wisconsin, and to invite the Wisconsin Association of Physics Teachers to participate in making program plans.

At the morning session the following papers were presented:

1. *Some aspects of the theory of loops*, by Professor R. H. Bruck, University of Wisconsin.

Professor Bruck discussed the notion of a "generalized ring" $R\equiv R(+, \cdot)$ forming a loop under addition, obeying the usual two-sided distributive law, and such that the non-zero elements of R form a group G under multiplication. He showed that, when the group G is given, the existence of R is equivalent to the existence of a permutation S of G with certain prescribed properties; and that addition in R may be defined in terms of S. Conditions were exhibited necessary and sufficient that R be the basis of a coördinate system for a projective geometry. The discourse concluded with an explanation of the known one-

to-one correspondence between finite projective geometrics with $n+1$ points on a line and complete sets of $(n-1)$ mutually orthogonal n-rowed latin squares.

2. *A brief survey of meteorology*, by W. R. Jarmain, introduced by the Secretary.

The speaker pointed out that the development of meteorology as applied to weather forecasting has been slow, due to the fact that it required some quite recently invented scientific devices. Not until the twentieth century was the subject on a mathematical basis, and even today, forecasting is still very much an art as well as a science. The importance of Bjerknes' polar front theory in making day to day predictions was emphasized. Five day forecasting is still in its infancy, and considerable research remains to be done in this field. Extra-terrestrial influences on the earth's weather were deemed rather unimportant. The speaker indicated that meteorology has made great strides in recent years, and has a bright future.

3. *Social implications of atomic energy*, by Professor Elda Anderson, Milwaukee Downer College, introduced by Professor Ethelwynn R. Beckwith.

At the afternoon session reports were given by members of the educational committee referred to above, indicating some aspects of the status of mathematical education in Wisconsin and activities of the National Committee for the Co-ordination of Studies in Mathematical Education. There followed a discussion in which there was general and enthusiastic participation. It was agreed that the Wisconsin Section should proceed to organize and plan for the purpose of cooperating in all ways feasible in a constructive effort to meet the problems of mathematical education in Wisconsin's secondary schools.

PAUL L. TRUMP, *Secretary*

ANNUAL MEETING OF THE METROPOLITAN NEW YORK SECTION

The fifth annual meeting of the Metropolitan New York Section of the Mathematical Association of America was held at The Cooper Union, Cooper Square, New York, N. Y., on Saturday, May 4, 1946. Professor H. E. Wahlert, Vice-Chairman of the Section, presided at the morning session; and Professor H. E. Miller, Chairman of the Section, presided at the afternoon session.

There were ninety present, including the following fifty-four members of the Association: R. G. Archibald, Brother Bernard Alfred (Welch), C. B. Boyer, A. D. Bradley, Benjamin Braverman, A. B. Brown, Jewell Hughes Bushey, H. R. Cooley, W. H. H. Cowles, W. H. Fagerstrom, Edward Fleisher, R. M. Foster, Harriet M. Griffin, J. I. Griffin, George Grossman, Frank Hawthorne, R. V. Heath, Morris Hertzig, J. H. Hlavaty, Solomon Hurwitz, L. C. Hutchinson, F. W. John, R. A. Johnson, Aida Kalish, Herman Karnow, Edward Kasner, L. S. Kennison, E. R. Kiely, Nathan Lazar, C. H. Lehmann, Emanuel Levine, H. F. MacNeish, John Mandel, D. May Hickey Maria, Joseph Milkman, F. H. Miller, E. C. Molina, P. B. Norman, L. F. Ollmann, Water Prenowitz, Edward

Rayher, W. D. Reeve, S. G. Roth, Leila Rubashkin, Charles Salkind, Aaron Shapiro, Lao G. Simons, James Singer, E. R. Stabler, H. E. Wahlert, Israel Wallach, Alan Wayne, V. H. Wells, R. C. Yates.

At the opening of the morning session Dean George F. Bateman welcomed the Section to The Cooper Union. At the opening of the afternoon session a brief business meeting was held at which the following officers were elected for the coming year: Chairman, H. E. Wahlert, New York University; Vice-Chairmen, W. H. H. Cowles, Pratt Institute, and Morris Hertzig, Forest Hills High School; Secretary, C. B. Boyer, Brooklyn College; Treasurer, Aaron Shapiro, Midwood High School.

The program consisted of the following papers:

1. *Cartesian geometry from Fermat to Lacroix*, by Professor C. B. Boyer, Brooklyn College.

The basic principle of coördinate geometry was discovered independently by Descartes and Fermat, but the works of these men differ markedly in emphasis. The speaker therefore proposed that the problem of finding the equations of given loci be designated "analytic geometry in the sense of Descartes," and that the inverse aspect, the study of curves determined by given equations, be referred to as "analytic geometry in the sense of Fermat." It was pointed out that the Cartesian aspect dominated thought for well over a century, but that the point of view of Fermat was represented in England by Newton and Maclaurin and on the Continent by Euler and Cramer. The widely-held opinion that Descartes arithmetized geometry was shown to be inconsistent with the Cartesian goal of exhibiting the geometric "constructibility" of determinate and indeterminate equations. A true arithmetization of geometry was hinted at by Lagrange, but it was finally carried out in 1795–1798 by Monge and Lacroix. Comparing this "analytical revolution" in France with the so-called chemical revolution of the same period, the speaker proposed that the idea of expressing a geometry in algebraic language be designated "analytic geometry in the sense of Monge and Lacroix."

2. *Geometry of ship waves*, by Professor J. J. Stoker, New York University, introduced by Professor H. E. Wahlert.

3. *Evaluating a syllabus in experimental geometry a priori*, by Charles Salkind, Samuel J. Tilden High School.

To replace the present tenth-year course, a group of New York City teachers of mathematics have prepared an instrument titled, "Experimental (non-Regents) Geometry Course." The speaker pointed out that the alleged superiority of the course lies in (1) exhibiting geometry as an elementary illustration of the scientific method (induction); (2) clarifying the complementary roles of experimentation and deduction. He held, however, that an a priori evaluation of the course discloses that it adds to existing burdens without removing any of the fundamental difficulties; and he questioned the validity of the proposed

program for the following reasons. (1) Procedures based *primarily* on measurement are amply provided for in the junior high school. (2) The results of deductive reasoning do *not* gain credence from experimentation and measurement. Contrary evidence exists to show that overindulgence in informal methods *inhibits* the acceptance of deductive results. (3) Scientific induction should be taught to secondary pupils, but by teachers specially qualified to do so. (4) The proposed course fails to meet the standards of a properly-planned educational experiment.

4. *Coordinating high school and college mathematics*, by Professor W. D. Reeve, Teachers College, Columbia University.

Professor Reeve's paper appears as the first article in the present issue.

5. *Infinity in art*, by Professor Edward Kasner, Columbia University.

The speaker described the construction and properties of what he called an *infinite Christmas tree*. From the upper end of a vertical unit line segment, two mutually perpendicular segments of length $\frac{1}{2}$ are constructed to make angles of 135° with the unit segment; from the extremities of each of these, two new mutually perpendicular segments of length $\frac{1}{4}$ are similarly drawn at 135° to the half-unit segments; and so on ad infinitum. The distance from the base to any one of the asymptotic limiting points, measured along the trunk and branches, is two units. The class of nodes (or branching points) is denumerable and hence has as its number Cantor's $\aleph_0$; the points of condensation (terminal buds) are non-denumerable and their number is that of the continuum c. The speaker suggested the determination of the area of the smallest circumscribing rectangle, the study of the design of nodes and buds, and the construction of an analogous configuration in three dimensions.

C. B. BOYER, *Secretary*

CALENDAR OF FUTURE MEETINGS

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN
ILLINOIS, Peoria, May 9–10, 1947
INDIANA
IOWA, Cedar Falls, April 18–19, 1947
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK, Brooklyn, April, 1947
MICHIGAN
MINNESOTA
MISSOURI
NEBRASKA, Lincoln, May 3, 1947
NORTHERN CALIFORNIA, San Francisco, January 25, 1947
OHIO, Columbus, April 3, 1947
OKLAHOMA
PACIFIC NORTHWEST
PHILADELPHIA
ROCKY MOUNTAIN
SOUTHEASTERN, Columbia, S. C., April 18–19, 1947
SOUTHERN CALIFORNIA, Claremont, March 8, 1947
SOUTHWESTERN
TEXAS
UPPER NEW YORK STATE, Rochester, May 10, 1947
WISCONSIN, Madison, May, 1947

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No. 4. *Projective Geometry,* by PROFESSOR J. W. YOUNG. (First Impression, 1930; Second Impression, 1938.)

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